Conjugate deficiency in finite groups

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Abstract. We consider the function \( r(G) = |G| - \kappa(G) \), where the group \( G \) has exactly \( \kappa(G) \) conjugacy classes. We find all \( G \) where \( r(G) \) is small and pose a number of relevant questions.

1. Introduction

Let \( G \) be a finite group and let \( G \) have exactly \( \kappa(G) \) conjugacy classes of elements. One of the most startling results in finite group theory is the following beautiful theorem of Burnside [3, p.295].

Theorem A. If \( |G| \) is odd, then \( |G| - \kappa(G) \equiv 0 \pmod{16} \).

We note that no such result can hold if \( |G| \) is even. For example, if \( S_3 \) is the symmetric group of order 6 and \( A_4 \) is the alternating group of order 12, then \( \kappa(S_3) = 3, \kappa(A_4) = 4 \), so that \( r(S_3) = 3, r(A_4) = 8 \), and \( \gcd(3, 8) = 1 \).

Burnside proved Theorem A using matrix representation theory, but later authors such as Hirsch [5] and Poland [7] proved Burnside’s result by elementary means and in fact generalized it. Theorem A has some immediate consequences which are pretty and useful enough to impress students taking a first course in group theory.

Consequence B. Groups of orders 3, 5, 7, 9, 11, 13, 15, and 17 are all abelian.

Consequence C. A non-abelian group of order 21 has exactly 5 conjugacy classes.

The form of Theorem A suggests that it would be worthwhile to consider the function \( r(G) := |G| - \kappa(G) \), which we call the conjugate deficiency of a finite group \( G \). In this note, we prove a number of results about \( r(G) \) including the following.

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Theorem 1. There are only finitely many groups $G$ with a given value of $r(G) > 0$.

We note the obvious fact that there are infinitely many finite groups with $r(G) = 0$, and these are precisely the abelian groups. In what follows, we disregard these groups, so that throughout $G$ will denote a finite non-abelian group.

We use the following notation for some families of groups: $C_n$ is the cyclic group of order $n$; $S_n$ is the symmetric group of order $n!$; $A_n$ is the alternating group of order $n!/2$; $D_n$ is the dihedral group of order $2n, n > 2$; and $Q_n$ is the dicyclic group of order $4n, n > 1$ (in particular, $Q_2$ is the quaternion group).

Theorem 2. There is no $G$ with $r(G) = 1, 2, 4, 5, \text{ or } 7$.

Theorem 3. The groups with $r(G) = 3$ are $S_3, D_4, \text{ and } Q_2$.

Theorem 4. There are exactly nine groups with $r(G) = 6$.

Theorem 5. The only group with $r(G) = 8$ is $A_4$.

This example $A_4$ knocks on the head the conjecture that $r(G) \equiv 0 \pmod{3}$ if $|G|$ is even. However Hirsch [5] shows that if $|G|$ is even and $3 \nmid |G|$, then $r(G) \equiv 0 \pmod{3}$. Also if $|G|$ is odd and $3 \nmid |G|$, then $r(G) \equiv 0 \pmod{48}$.

Theorem 6. The odd order groups which satisfy $r(G) = 16$ are one group of order 21 and two groups of order 27.

Theorem 7. The only odd order group which satisfies $r(G) = 32$ is the non-abelian group of order 39.

Theorem 8. There are exactly six odd order groups satisfying $r(G) = 48$.

We begin with the following elementary lemma which, combined with a knowledge of groups of small order, yields all the above results.

Lemma 9. Suppose $G$ is a non-abelian group. Let $p$ be the least prime dividing $|G|$, and suppose $(G : Z(G)) \geq n$, where $Z(G)$ is the centre of $G$. Then

$$k(G) \leq \frac{n + p - 1}{pn} \cdot |G|.$$
In particular,

\[ k(G) \leq \frac{p^2 + p - 1}{p^3} \cdot |G|. \]

Proof. The number of single element conjugacy classes in \( G \) equals \( |Z(G)| \), and so is at most \( |G|/n \). Since the size of a conjugacy class is a divisor of \( |G| \), any other class has at least \( p \) elements, so

\[ k(G) \leq \frac{1}{n} |G| + \frac{1}{p} \left( 1 - \frac{1}{n} \right) |G| = \frac{n + p - 1}{pn} \cdot |G|. \]

Since \( G \) is non-abelian, \( G/Z(G) \) is not cyclic. Thus we can take \( n = p^2 \) to get the second estimate.

We remark that this result is best possible, being attained for the non-abelian groups of order \( p^3 \), both for \( p = 2 \) and \( p \) an odd prime. It follows from Lemma 9 that

\[ r(G) = |G| - k(G) \geq |G| \left( 1 - \frac{n + p - 1}{np} \right) = \frac{(n-1)(p-1)}{np} |G|. \]

Thus

\[ |G| \leq \frac{np \cdot r(G)}{(n-1)(p-1)}, \tag{1} \]

where \( p \) is the least prime dividing \( |G| \) and \( n \leq (G: Z(G)) \). Using the second estimate in Lemma 9, we get

\[ |G| \leq \frac{p^3 \cdot r(G)}{(p^2 - 1)(p - 1)}, \tag{2} \]

Since \( p^3/(p^2 - 1)(p - 1) \) obviously decreases as \( p \) increases, we have the following:

\[ |G| \leq \frac{8r(G)}{3}, \quad \text{for all finite non-abelian groups } G. \tag{3} \]

\[ |G| \leq \frac{27r(G)}{16}, \quad \text{for all finite non-abelian groups } G \text{ of odd order.} \tag{4} \]

Moreover, we have equality in (3) if and only if \((G : Z(G)) = 4\), and equality in (4) if and only if \((G : Z(G)) = 9\). By (3), there is an upper bound on \( |G| \) for any given \( r(G) > 0 \). Theorem 1 now follows since there are only finitely many finite groups whose order does not exceed a given number.

Using (3), we see that \(|G| \leq 16/3 \) if \( r(G) \leq 2 \), and no such non-abelian group exists. If \( r(G) = 3 \), then \(|G| \leq 8 \). There are exactly 3
non-abelian groups of order at most 8, namely $S_3$, $D_4$ and $Q_2$, and $r(G) = 3$ in all three cases.

Using (3), we see that $|G| \leq 16$ if $r(G) \leq 6$, so to understand how $4 \leq r(G) \leq 6$ can arise, we need to examine all non-abelian groups of orders between 9 and 16 inclusive. There are fourteen such groups, and for nine of these we have $k(G) = 6$, namely $D_5$; $Q_3$; $D_6 = S_3 \times C_2$; and the six groups of order 16 with $(G : Z(G)) = 4$, namely $D_4 \times C_2$, $Q_2 \times C_2$, and $16/8, 16/9, 16/10, 16/11$, in the notation of [8]. The five remaining non-abelian groups with orders between 9 and 16 inclusive have larger deficiencies: $k(A_4) = 8$ and $k(D_7) = k(D_8) = k(Q_4) = k(SD_{16}) = 9$, where $SD_{16}$ is the semidihedral group of order 16. Thus there are no groups with $r(G) \in \{4, 5\}$, and nine groups with $r(G) = 6$.

Using (3), we see that $|G| \leq 64/3$ if $r(G) \leq 8$, so to understand how $7 \leq r(G) \leq 8$ can arise, we need to examine the five non-abelian groups with order at most 16 and $r(G) > 6$, plus groups of order between 17 and 21 inclusive. Of the five with order at most 16 and $k(G) > 6$, the only one with $r(G) \leq 8$ is $A_4$ giving $r(A_4) = 8$.

As for the groups of larger order between 17 and 21, we need only check the even order groups, since (4) tells us that $|G| \leq 27/2 < 16$ if $|G|$ is odd and $r(G) \leq 8$. It remains to check $|G| \in \{18, 20\}$, and there are six such groups: three of order 18 ($D_9$, $S_3 \times C_3$, and a semidirect product of $C_3 \times C_3$ by $C_2$) and three of order 20 ($D_{10}$, $Q_5$, and the general affine group of degree 1 over GF$_5$). In each case, $r(G) > 8$. This establishes Theorems 2, 3, 4, and 5.

We now turn to the case where $G$ has odd order, as suggested by Theorem A. If $|G|$ is odd and $r(G) = 16$, then by (4), $|G| \leq 27$, and just three groups emerge: the non-abelian group of order 21, and two groups of order 27. Again for $|G|$ odd and $r(G) = 32$, we must have $|G| \leq 54$ and just one group emerges, namely the non-abelian group of order 39. For $|G|$ odd and $r(G) = 48$, we must have $|G| \leq 81$, and we get 10 groups: one of order 55, one of order 57, two of order 63, and six of order 81. This establishes Theorems 6, 7, and 8.

Now let $t(n)$ be the number of groups which satisfy $r(G) = n$. Here is a table listing the values of $t(n)$ for $n \leq 30$, obtained by the above methods.
The dihedral groups alone suffice to get $r(G)$ equal to any multiple of 3. In fact for $n > 1$, it is well known that $k(D(2n - 1)) = n + 1$ and $k(D(2n)) = n + 5$, and so $r(D(2n - 1)) = r(D(2n)) = 3(n - 1)$.

It seems difficult to predict the values of $t(n)$, but it is easy to see that

$$r(A \times G) = |A|r(G)$$

whenever $A$ is a finite abelian group. Since there are abelian groups of all orders, it follows that if a given number $n$ is a value of $r(G)$, then so is $mn$ for all $m \in \mathbb{N}$. Moreover $t(mn) \geq t(n)$ for all $m, n \in \mathbb{N}$. This suggests that it would be important to consider prime numbers $p$ for which $t(p) > 0$.

We note that for each prime $p$, there is a group of order $p^3$ with $p^2 + p - 1$ classes, so that $r(G) = (p^2 - 1)(p - 1)$ is always possible. In addition, if $p$ and $q$ are primes with $2 < p < q$, where $p \mid (q - 1)$, then the nonabelian group of order $pq$ has $p + (q - 1)/p$ conjugacy classes, and so

$$r(G) = \frac{(q - 1)(p^2 - 1)}{p}.$$

We close with a number of related problems, some of which could prove difficult to solve.

**Problem 1.** Give a realistic upper bound for $t(n)$ for each $n$.

**Problem 2.** Characterize the numbers $n$ for which $t(n) = 0$.

With the help of [8] and GAP [4], we see that the numbers in the above problem begin

$1, 2, 4, 5, 7, 10, 11, 13, 14, 20, 22, 23, 25, 26, 28, 29, 31, 37, 41, 43, 46, 47, 49, 50, 52, 53, 58, 59, 61, 62, \ldots$

Are there infinitely many such numbers?

**Problem 3.** Are there infinitely many primes $p$ for which $t(p) > 0$?
The primes less than 199 for which \( t(p) > 0 \) are as follows:

\[
3, 17, 19, 83, 97, 107, 113, 137, 149, \\
151, 157, 167, 173, 179, 181, 193, 197.
\]

These values were found using the Small Groups Library of GAP ([4], [1]) by searching through groups of order at most 511.

**Problem 4.** Are there infinitely many pairs \((n, n+1)\) where \(3 \nmid n\) and \(3 \nmid (n+1)\) such that \(t(n) = t(n+1) = 0\)?

**Problem 5.** For each \(k \geq 4\), is there an odd order group \(G\) with \(r(G) = 2^k\)?

If the answer to this last problem is positive, then we can find a group of odd order with \(r(G) = 16l\) for all \(l \in \mathbb{N}\) by taking direct products as previously described. The answer is indeed positive for \(4 \leq k \leq 12\), because of groups of order 21, 39, 75, 147, 291, 579, 1161, 2307, 4221; the largest three of these orders were found with the help of GAP. The desired group is given in all except two cases by a semidirect product \(C_n \rtimes C_3\), for \(n = |G|/3\). The two exceptional cases are \(|G| = 75\) in which case \(G = C_5^2 \rtimes C_3\), and \(|G| = 4221\) in which case \(G\) is of type \((C_7 \times C_3) \times (C_{67} \times C_3)\). There does not seem to be a clear enough pattern to these examples to justify a conjecture that the answer is always positive.

**Problem 6.** Is the function \(t(n)\) onto \(\mathbb{N}\)? Is there, for example, an \(n\) with \(t(n) = 2\)?

**Problem 7.** For \(n\) odd and \(n > 3\), do there exist primes \(p\) and \(q\) with \(2 < p < q\) where \(p \mid (q - 1)\), such that \(n = p + (q - 1)/p\)?

Computer results [2] show that this result is true for all \(n, 3 < n < 10000001\). If it is true in general, then it provides an answer to the following question posed by the second author in [6].

For each odd \(k > 3\), does there exist an odd order non-abelian group with exactly \(k\) conjugacy classes?

Of particular interest is \(r(S_n) = n! - p(n)\), where \(p(n)\) is the number of partitions of \(n\). This purely arithmetic function is of some interest in its own right, so we ask:

**Problem 8.** What is the range of values of \(r(S_n)\)?

We say that \(n\) is primitive if \(t(n) \neq 0\), but \(t(d) = 0\) for each proper divisor \(d\) of \(n\). For example, 3, 8, 17, and 19 are primitive.
Problem 9. Are there infinitely many primitive values of $n$?

REFERENCES

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