Irish Math. Soc. Bulletin
Number 70, Winter 2012, 69–72
ISSN 0791-5578

PROBLEMS

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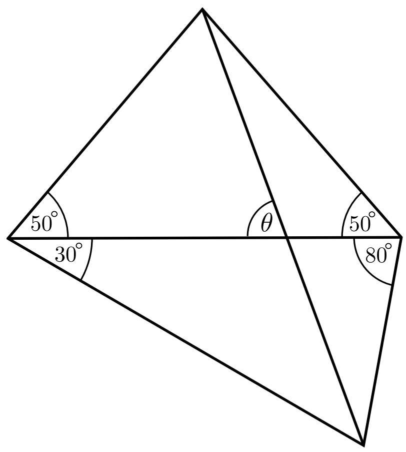
Problems

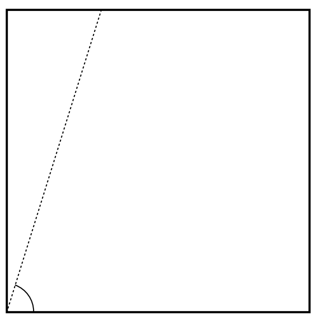
First a geometry problem, which I was told a few years ago.

Problem 70.1. Find θ.



The second problem, known to many of you, is about trisecting an angle using origami. Resist the temptation to find a solution on the internet!

Problem 70.2. Fold a square piece of paper to create an angle, as shown below.



Using a sequence of folds, trisect this angle.

Received on 18-1-2013.

The final problem was contributed by Finbarr Holland.

Problem 70.3. Let n be a positive integer. Prove that

$$\sum_{m=-\infty}^{\infty}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in|x|}e^{-imx}\,dx\right| = 1+\frac{4}{\pi}\sum_{k=1}^{n}\frac{1}{2k-1}\,.$$

Solutions

Here are the solutions to the problems from *Bulletin* Number 68. The solution to the first problem was contributed by the North Kildare Mathematics Problem Club. This problem was also solved by the proposer.

Problem 68.1. A polynomial is said to be *stable* if all its roots have negative real part. Suppose that

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \qquad a_n \neq 0,$$

is stable. Prove that

$$q(z) = a_{n-1}z^{n-1} + 2a_{n-2}z^{n-2} + \cdots + (n-1)a_1 z + na_0$$

is also stable.

Solution 68.1. Let

$$f(z) = z^n p\left(\frac{1}{z}\right).$$

Then the roots of f are reciprocals of the roots of p, so f is stable. The Gauss-Lucas theorem tells us that the roots of f' lie in the convex hull of the roots of f, so f' is also stable. Now observe that

$$q(z) = z^{n-1}f'\left(\frac{1}{z}\right).$$

It follows that the roots of q are reciprocals of the roots of f', so q is stable too. □

The solution to the second problem is due to Prithwijit De from the Homi Bhabha Centre for Science Education, Mumbai, India. This problem was also solved by the North Kildare Mathematics Problem Club and the proposer.

Problem 68.2. Let A denote the set of positive integers that do not contain a 9 in their decimal expansion. Determine whether the sum

$$\sum_{n\in A}\frac{1}{n}$$

converges or diverges.

Solution 68.2. Let S_m denote the set of integers in A with m digits. These sets partition A, therefore

$$\sum_{n\in A}\frac{1}{n}=\sum_{m=1}^{\infty}\sum_{n\in S_m}\frac{1}{n}.$$

If $n\in S_m$ then $n\geqslant 10^{m-1}$. Also, $|S_m|=8\times 9^{m-1}$. Therefore

$$\sum_{m=1}^{\infty}\sum_{n\in S_m}\frac{1}{n}\leqslant\sum_{m=1}^{\infty}\frac{8\times 9^{m-1}}{10^{m-1}}=80.$$

It follows that $\sum_{n\in A}1/n$ converges. □

The solution to the third problem was contributed by the North Kildare Mathematics Problem Club. This problem was also solved by the proposer.

Problem 68.3. Given subsets U and V of a finite group G, define

$$UV=\{uv\,:\,u\in U,v\in V\}$$

and

$$U^{-1}=\{u^{-1}\,:\,u\in U\}.$$

Prove that

$$|AB||A^{-1}A\cap BB^{-1}|\geqslant |A||B|,$$

for any pair of subsets A and B of G.

Solution 68.3. The inequality is true if either A or B are empty, so let us suppose that neither is empty. Consider the surjective function

$$\phi : A\times B\longrightarrow AB,\quad (a,b)\longmapsto ab.$$

Then

$$\phi^{-1}(ab)=\left\{(ac,c^{-1}b):c\in A^{-1}A\cap BB^{-1}\right\}\cap A\times B.$$

To see this, suppose that $(a_0,b_0)\in\phi^{-1}(ab)$, so that $(a_0,b_0)\in A\times B$ and $a_0b_0=ab$. Let $c=a^{-1}a_0=bb_0^{-1}$, an element of $A^{-1}A\cap BB^{-1}$. Then $(a_0,b_0)=(ac,c^{-1}b)$. The reverse inclusion is straightforward.

It follows that

$$|A^{-1}A\cap BB^{-1}|\geqslant|\phi^{-1}(ab)|.$$

Because the sets $\phi^{-1}(ab)$ partition $A\times B$, we conclude that

$$|AB||A^{-1}A\cap BB^{-1}|\geqslant |A||B|.$$

□

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com`.

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