Irish Math. Soc. Bulletin
Number 69, Summer 2012, 25–32
ISSN 0791-5578

"ODD" MATRICES AND EIGENVALUE ACCURACY

DAVID JUDGE

Abstract. A definition of *even* and *odd* matrices is given, and some of their elementary properties stated. The basic result is that if λ is an eigenvalue of an odd matrix, then so is $-\lambda$. Starting from this, there is a consideration of some ways of using odd matrices to test the order of accuracy of eigenvalue routines.

1. Definitions and some elementary properties

Let us call a matrix W **even** if its elements are zero unless the sum of the indices is even – i.e. $W_{ij} = 0$ unless $i + j$ is even; and let us call a matrix B **odd** if its elements are zero unless the sum of the indices is odd – i.e. $B_{ij} = 0$ unless $i + j$ is odd. The non-zero elements of W and B (the letters W and B will always denote here an even and an odd matrix, respectively) can be visualised as lying on the white and black squares, respectively, of a chess board (which has a white square at the top left-hand corner).

Obviously, any matrix A can be written as $W+B$; we term W and B the *even* and *odd* parts, respectively, of A. Under multiplication, even and odd matrices have the properties, similar to those of even and odd functions, that

$$\begin{array}{ccccc} \textit{even} \times \textit{even} & \text{and} & \textit{odd} \times \textit{odd} & \text{are} & \textit{even}, \\ \textit{even} \times \textit{odd} & \text{and} & \textit{odd} \times \textit{even} & \text{are} & \textit{odd}. \end{array}$$

From now on, we consider only **square** matrices. It is easily seen that, if it exists, the inverse of an even matrix is even, the inverse of an odd matrix is odd. It is also easily seen that in the PLU decomposition of a non-singular matrix A which is either even or odd, L and U are always even, while P is even or odd according as A is.

2010 *Mathematics Subject Classification.* 15A18,15A99,65F15.
Key words and phrases. even matrix, odd matrix, eigenvalues, test matrix.
Received on 27-4-2012; revised 11-7-2012.
The author would like to express his gratitude for helpful discussions with T. Laffey, D. Lewis, and the late Fergus Gaines.

We note also that in the QR factorisation of a non-singular even matrix, R and Q are both even, while for an odd matrix, R is even, Q is odd. (This is most easily seen by viewing the QR factorisation in its Gram-Schmidt orthogonalisation aspect.) Thus the property of being even or odd is preserved under the basic QR algorithm for eigenvalues: however, for odd matrices, this is not true for the QR algorithm with shifts.

2. The basic result

The following very elementary result is basic:

Proposition 2.1. *If λ is an eigenvalue of an odd matrix B, then so is $-\lambda$.*

This can be seen easily in two different ways.

First Proof. If $D_{jk} = (-1)^k \delta_{jk}$, $D^{-1}(W+B)D = W - B$, so that $W+B$, $W-B$ have the same eigenvalues, for arbitrary square (even and odd) matrices W and B. Putting $W = 0$, the result follows. $\square$

Second Proof. We can write any vector $\mathbf{x}$ as $\mathbf{x} = \mathbf{u} + \mathbf{v}$, where $u_i = 0$, i odd, $v_i = 0$, i even. We call $\mathbf{u}$ and $\mathbf{v}$ *even* and *odd* vectors, respectively, and refer to them as the *even* and *odd* parts of $\mathbf{x}$. (We assume that indices run from 1 to n, the order of the matrix. However, choosing the index origin as 0 merely interchanges the values of $\mathbf{u}$ and $\mathbf{v}$, and makes no difference to what follows.) We note that

$$if\ B\ is\ odd,\quad then\quad B\mathbf{u}\ \ is\ \ odd,\ \ B\mathbf{v}\ \ is\ \ even. \tag{1}$$

Now if $\mathbf{x}$ is an eigenvector of B with eigenvalue λ,

$$B\mathbf{x} = \lambda\mathbf{x}, \tag{2}$$

writing $\mathbf{x} = \mathbf{u} + \mathbf{v}$, and using (1), we must have

$$B\mathbf{u} = \lambda\mathbf{v}\ \ , \qquad B\mathbf{v} = \lambda\mathbf{u}. \tag{3}$$

Then

$$B(\mathbf{u}-\mathbf{v}) = B\mathbf{u} - B\mathbf{v} = \lambda\mathbf{v} - \lambda\mathbf{u} = -\lambda(\mathbf{u}-\mathbf{v}), \tag{4}$$

so that $-\lambda$ is an eigenvalue of B, with eigenvector $\mathbf{u}-\mathbf{v}$. $\square$

This property is useful in giving a simple indication of the accuracy of eigenvalue routines. For a completely arbitrary odd matrix B, random or otherwise, with real or complex elements, the non-zero eigenvalues can be sorted into pairs $(\lambda_i^1, \lambda_i^2)$ such that

$$\lambda_i^1 + \lambda_i^2 = 0. \tag{5}$$

(As far as this property is concerned, there is no initial error whatever in entering B .) If the computed values, denoted by $\hat{\lambda}$, are sorted into corresponding pairs $(\hat{\lambda}_i^1, \hat{\lambda}_i^2)$, and

$$\delta_i \equiv \hat{\lambda}_i^1 + \hat{\lambda}_i^2 \ , \tag{6}$$

then $\max|\delta_i|$ gives an estimate of the error.

A check on the accuracy of the eigenvectors is also possible. The argument of proof 2 shows that if $\mathbf{x} = \mathbf{u} + \mathbf{v}$ is an eigenvector of B corresponding to a simple eigenvalue λ , and if $\mathbf{y}$ is an eigenvector corresponding to $-\lambda$, then $\mathbf{y}$ is of the form $\mathbf{y} = \theta\mathbf{u} + \phi\mathbf{v}$, with the *same* $\mathbf{u}$ and $\mathbf{v}$, where θ and ϕ are scalars. (The fact that $\phi = -\theta$ here is not essential.) Thus if we multiply $\mathbf{y}$ by a factor to make one of its even components (say the first non-zero one, or the largest one) equal to the corresponding component of $\mathbf{x}$, producing $\mathbf{z}$, say, we must have $\mathbf{z} = \mathbf{u} + \rho\mathbf{v}$, where ρ is some scalar. When we subtract $\mathbf{z}$ from $\mathbf{x}$, all the other even components must also cancel exactly. Doing this for the computed vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, the actual deviations from zero of these components give an accuracy estimate. The odd components can be checked similarly.

3. Even matrices

For even matrices, there is no necessary relation whatever between any of the eigenvalues – e.g., a diagonal matrix with arbitrary values is even.

However, it is of interest to consider a further subdivision of such matrices into a *doubly-even* (in brief, d-even) part and a *doubly-odd* (in brief, d-odd) part,

$$W = W_{ee} + W_{oo}, \tag{7}$$

where $(W_{ee})_{ij} = 0$ unless i and j are both *even*, $(W_{oo})_{ij} = 0$ unless i and j are both *odd*.

Under multiplication, doubly-even and doubly-odd matrices have the properties that

$$\begin{array}{llllll} \textit{d-even} \times \textit{d-even} & \text{is} & \textit{d-even}, & \textit{d-odd} \times \textit{d-odd} & \text{is} & \textit{d-odd}, \\ \textit{d-even} \times \textit{d-odd} & \text{is} & \textit{zero}, & \textit{d-odd} \times \textit{d-even} & \text{is} & \textit{zero}. \end{array}$$

Thus the doubly-even and doubly-odd parts are completely 'uncoupled' under multiplication.

If I denotes the unit matrix, let I_{ee} and I_{oo} denote its doubly-even and doubly-odd parts, so that

$$I = I_{ee} + I_{oo}. \tag{8}$$

We can now see that the d-even and d-odd parts are also 'uncoupled' when taking inverses, in the following sense. If W is non-singular, let X and Y denote the d-even and d-odd parts, respectively, of W^{-1}. Then

$$X \times W_{ee} = I_{ee}, \quad X \times W_{oo} = 0, \quad Y \times W_{ee} = 0, \quad Y \times W_{oo} = I_{oo}.$$

The eigenvalue problem for W also splits into two completely independent sub-problems:

$$W\mathbf{x} = \lambda\mathbf{x} \quad \Longrightarrow \quad W_{ee}\mathbf{u} = \lambda\mathbf{u}, \; W_{oo}\mathbf{v} = \lambda\mathbf{v},$$

where $\mathbf{u}$ and $\mathbf{v}$ are the even and odd parts of $\mathbf{x}$.

These properties of even matrices, trivial to check, are even more obvious on noting that under a simple reordering transformation, W is equivalent to a block-diagonal matrix of the form

$$T = \left[\begin{array}{c|c} T1 & 0 \\ \hline 0 & T2 \end{array}\right] \tag{9}$$

where $T1$, of dimension $\lfloor (n+1)/2 \rfloor$, contains the non-zero elements of W_{oo}, and $T2$, of dimension $\lfloor n/2 \rfloor$, contains those of W_{ee}.

We note that under the same transformation, an odd matrix B is similarly equivalent to a skew-diagonal block matrix

$$S = \left[\begin{array}{c|c} 0 & S1 \\ \hline S2 & 0 \end{array}\right]. \tag{10}$$

Thus the determinant of an even matrix always factorizes,

$$\det(W) = \det(T) = \det(T1).\det(T2),$$

and the determinant of an odd matrix either factorizes or is zero:

$$\begin{aligned} \det(B) = \det(S) &= (-1)^{n/2}\det(S1).\det(S2), \text{ if } n \text{ is even}, \\ &= 0 \text{ if } n \text{ is odd} \end{aligned}$$

(since if n is odd, $S1$ and $S2$ are not square). This latter gives one proof of the almost obvious fact that an odd matrix of odd dimension always has the eigenvalue $\lambda = 0$.

4. Further eigenvalue matchings

We consider now the matrix

$$M \ \equiv \ \alpha I_{ee} + \beta I_{oo} + B \tag{11}$$

where α and β are scalars, B as usual denotes an odd matrix, and I_{ee} and I_{oo} are as defined above at (8). If λ is an eigenvalue of M, the eigenvalue equation

$$M\mathbf{x} = \lambda\mathbf{x} \tag{12}$$

can be written as the pair of equations

$$\alpha\mathbf{u} + B\mathbf{v} = \lambda\mathbf{u}, \tag{13}$$

$$B\mathbf{u} + \beta\mathbf{v} = \lambda\mathbf{v} \tag{14}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the even and odd parts of $\mathbf{x}$. We can now proceed in two different ways.

First, on defining $\mathbf{y}$ by

$$\mathbf{y} = (\lambda - \alpha)\mathbf{u} + (\beta - \lambda)\mathbf{v} \tag{15}$$

we find that

$$M\mathbf{y} = (\alpha + \beta - \lambda)\mathbf{y}. \tag{16}$$

Thus the eigenvalues of the matrix M can be grouped into pairs (λ_1, λ_2) such that

$$\lambda_1 + \lambda_2 = \alpha + \beta. \tag{17}$$

This may be viewed as a generalisation of the basic result for a pure odd matrix B, where $\alpha = \beta = 0$.

Second, on defining $\mathbf{z}$ by

$$\mathbf{z} = \sqrt{(\lambda - \alpha)}\ \mathbf{u} + \sqrt{(\lambda - \beta)}\ \mathbf{v}, \tag{18}$$

we find that

$$B\mathbf{z} = \sqrt{\lambda - \alpha}\ \sqrt{\lambda - \beta}\ \mathbf{z}. \tag{19}$$

Thus for each eigenvalue λ of M, there corresponds an eigenvalue κ of B such that

$$\kappa = \sqrt{(\lambda - \alpha)(\lambda - \beta)}. \tag{20}$$

It should be emphasised that the relation (17) relates two eigenvalues of the *same* matrix, generated during *one* calculation, while

(20) relates eigenvalues of two *different* matrices, generated in *two independent* calculations.

Based on (20), we can give now an error estimate alternative to (6). It is convenient to take $\beta = -\alpha$, and solve for $\lambda, \lambda = \sqrt{(\kappa^2 + \alpha^2)}$, so that the sets of λ's and κ's can be sorted into pairs (λ_i, κ_i) such that

$$\lambda_i - \sqrt{(\kappa_i^2 + \alpha^2)} = 0. \tag{21}$$

If the computed values, denoted by $\hat{\lambda}, \hat{\kappa}$, are sorted into corresponding pairs $(\hat{\lambda}_i, \hat{\kappa}_i)$, and

$$\delta_i \equiv \hat{\lambda}_i - \sqrt{(\hat{\kappa}_i^2 + \alpha^2)}\ , \tag{22}$$

then again $\max|\delta_i|$ gives an estimate of the error.

We note that for any estimate based on either (17) or (as here) (20), one can check on the eigenvectors also, just as in Section 2.

5. Discussion

If one is using a package to calculate eigenvalues and wishes to get some idea of the accuracy of the results, it is natural to see how it performs on *test matrices.* An ideal test matrix is one which can be entered exactly, and whose eigenvalues are known in advance, either exactly or to very high precision. The error level can thus be assessed directly by observing the differences between the computed values and the correct values.

Rather than importing samples of this type, one may prefer to use something which is easily generated. For example, a simple method is to choose a set of numbers and form a diagonal matrix D, say, with these values; then choose an arbirary non-singular matrix X, say, and take $M \equiv XDX^{-1}$ as the test matrix. Its eigenvalues should be the chosen numbers, and its eigenvectors the columns of X. However, there is a snag: what is entered into the eigenvalue calculation is not, in fact, the exact 'known-eigenvalue' matrix M, but a computed approximation, $\hat{M}$, say; and so one cannot tell how much of the observed error is due to the eigenvalue calculation, and how much arises in computing $\hat{M}$.

An alternative approach, the one adopted here, is to use an exact-entry test matrix, or pair of matrices, whose eigenvalues are *not* known in advance: but, instead, some *relation* which should hold between these values is known. The extent to which the the computed values fail to satisfy this relation gives an estimate of the error due to the eigenvalue calculation alone.

However, caution is needed with this approach. Consider the following argument: "if λ is an eigenvalue of a *real* matrix, then so is $\bar{\lambda}$. Therefore we can sort the complex eigenvalues into pairs, the sum of whose imaginary parts should be zero. The difference from zero of the sum of the computed imaginary parts then gives an indication of the error of these computed values." This is, of course, completely erroneous, because the imaginary parts will be evaluated at the *same* point of the calculation as being $\pm$ the square-root of the *same* number,Q, say; and their sum will thus cancel perfectly, even though Q may differ widely from the correct value.

This shows that we must guard against the possibilty that the supposed 'accuracy-checking' relation between eigenvalues may be automatically satisfied at the time of their calculation. (One might worry that this could somehow be happening in (6), i.e. that the the 'equal-and-opposite' pairs might be produced in some correlated way because their magnitudes are equal. To circumvent this, one could add α times the unit matrix to B before finding the eigenvalues, and then subtract α from each before pairing - equivalent to (17) with $\beta = \alpha$. However, trials show that this step is unnecessary.) Clearly, there is no possiblity whatever of correlation between the errors of the $\kappa' s$ and $\lambda' s$ in (22).

For a *real* odd matrix B, if λ is an eigenvalue, then $\pm\lambda, \pm\bar{\lambda}$ are all in the list of eigenvalues, which complicates the sorting and matching of the imaginary parts. To keep (6) easy to implement, one can consider just the real parts, and simply ignore the imaginary parts of the λ's, implicitly assuming that the errors of the real and imaginary parts are of the same order. (This may be false in special circumstances, e.g. if B is antisymmetric – a case where, in fact, using (6) fails completely.) Using (22) has no such problems: we can take all the real parts and all the imaginary parts, of both the $\hat{\lambda}_i$ and the $\sqrt{(\hat{\kappa}_i^2 + \alpha^2)}$, as positive, and sort these real and imaginary parts independently.

In trials of random odd matrices (real and complex) of dimension up to 500, using (6)and (22) gave results of the same order, differing by a factor often close to one, and rarely exceeding two.

6. Addendum

The author is grateful to the referee for pointing out that very similar concepts have been introduced and used by Liu *et al* in the field of computer science: in [1], a square matrix A of even dimension n is decomposed into four sub-matrices, each of dimension $n/2$, containing those elements A_{ij} of A for which i and j are *even* and *even*, *even* and *odd*, *odd* and *even*, or *odd* and *odd*, respectively. (These correspond to the T_1, S_1, S_2, and T_2 of section 3, the index origin in [1] being 0.) These authors then exploit this partitioning to break down the process of evaluating $A \times \mathbf{b}$, where $\mathbf{b}$ is a vector of dimension n, into four processes involving the even and odd sub-vectors of $\mathbf{b}$ (each of dimension $n/2$), as a key step in designing an improved hardware module for use in vector digital signal processing.

If, motivated by this work, we consider a splitting, similar to 7, of an n-dimensional square odd matrix B into an "*even-odd*" part and an "*odd-even*" part (both n-dimensional),

$$B = B_{eo} + B_{oe}, \tag{23}$$

where $(B_{eo})_{ij} = 0$ unless i is *even*, j is *odd*, $(B_{oe})_{ij} = 0$ unless i is *odd*, j is *even*, it is of interest to note the easily-proved facts

(a) if λ is an eigenvalue of $B_{eo} + B_{oe}$, then $i\lambda$ is an eigenvalue of $B_{eo} - B_{oe}$,

(b) if W is an even matrix, the matrices $W, W + B_{eo}$ and $W + B_{oe}$ all have the same eigenvalues.

References

[1] Liu et al.: *Matrix odd-even partition: a high power-efficient solution to the small grain data shuffle.* Proc of the 6th IEEE International Conf on Networking, Architecture, and Storage. (2011) 348-54. (DOI 10.1109/NAS.2011.29)

David Judge was a member of the Mathematical Physics Department, University College Dublin, for many years until his retirement in 1996.

Mathematical Physics Department, University College Dublin
E-mail address: david.judge@ucd.ie