SPECTRAL PERMANENCE

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Abstract. Several kinds of generalized inverse bounce off one another in the proof of a variant of spectral permanence for C* embeddings.

This represents an expanded version of our talk to the IMS meeting of August 2012, which in turn was based on the work [3] of Dragan Djordjevic and Szezena Zivkovic of Nis, in Serbia.

1. Gelfand property

Spectral permanence, for C* algebras, says that the spectrum of an element \( a \in A \subseteq B \) of a C* algebra is the same whether it is taken relative to the subalgebra \( A \) or the whole algebra \( B \): this discussion is sparked by the effort to prove that the same is true of a variant of spectral permanence in which the two-sided inverse, whose presence or not defines “spectrum”, is replaced by a generalized inverse. The argument involves a circuitous tour through “group inverses”, “Koliha-Drazin inverses” and “Moore-Penrose inverses”; it turns out that the induced variants of spectral permanence are curiously inter-related.

Suppose \( T : A \rightarrow B \) is a semigroup homomorphism, where we insist that a semigroup \( A \) has an identity \( 1 \), and that a homomorphism \( T : A \rightarrow B \) respects that: we might indeed talk about a functor between categories. It then follows, writing \( A^{-1} \) for the invertible group in \( A \), that

\[
T(A^{-1}) \subseteq B^{-1},
\]

or equivalently, turning it inside out,

\[
A^{-1} \subseteq T^{-1}B^{-1}.
\]

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At its most abstract then “spectral permanence” for the homomorphism $T$ says that (1.2) holds with equality:

$$T^{-1}B^{-1} \subseteq A^{-1} \quad (1.3)$$

In words, it is tempting to describe (1.3) by saying “Fredholm implies invertible”. We shall also describe (1.3) as the Gelfand property, since it also holds, famously, when

$$T = \Gamma : A \to C(X) \subseteq C^X \quad (1.4)$$

is the Gelfand representation of a commutative Banach algebra $A$; here of course $X = \sigma(A)$ is the “maximal ideal space” of the algebra $A$. We might notice a secondary instance of spectral permanence in the embedding

$$C(X) \subseteq C^X \quad (1.5)$$

of continuous functions among arbitrary functions; similarly, for a Banach space $X$, the embedding

$$B(X) \subseteq L(X) \quad (1.6)$$

of bounded operators among arbitrary linear operators has spectral permanence, but only thanks to the ministrations of the open mapping theorem. Another elementary example is the left regular representation

$$L : A \to A^A \quad (1.7)$$

of the semigroup $A$ as mappings, where, for $a \in A$,

$$L_a(x) = ax \ (x \in A) \quad (1.8)$$

Less familiar is a commutant embedding

$$J : A = \text{comm}_B(K) \to B \quad (1.9)$$

where

$$\text{comm}_B(K) = \{b \in B : a \in K \implies ba = ab\} \quad (1.10)$$

and of course $J(a) = a$: here spectral permanence reflects the fact that two-sided inverses double commute:

$$a \in B^{-1} \implies a^{-1} \in \text{comm}_B^2(a) \quad (1.11)$$

If in particular the semigroup $A$ is a ring, having therefore a background “addition” and a distributive law, then we can quotient out the Jacobson radical

$$\text{Rad}(A) = \{a \in A : 1 - Aa \subseteq A^{-1}\} \quad (1.12)$$
in which every possible expression $1 - ca$ has an inverse: now it is easily checked that

$$K : a \mapsto a + \text{Rad}(A) \ (A \to A/\text{Rad}(A))$$

(1.13)

has spectral permanence. Our final example will be the most familiar, if not by any means the most elementary: it is the determinant

$$\det : \mathbb{C}^{n \times n} \to \mathbb{C},$$

(1.14)

which indeed “determines” whether or not a square matrix is invertible.

2. Spectral permanence

Mathematicians are thus prepared to go to a lot of trouble to establish spectral permanence. If we specialise to linear homomorphisms between (complex) linear algebras then we meet the phenomenon of spectrum, defining for each $a \in A$,

$$\sigma_A(a) = \{ \lambda \in \mathbb{C} : a - \lambda \notin A^{-1} \};$$

(2.1)

the idea is to harness complex analysis to the theory of invertibility. Now we can rewrite (1.1) to say that, for arbitrary $a \in A$,

$$\sigma_B(Ta) \subseteq \sigma_A(a),$$

(2.2)

while the Gelfand property (1.3) says that (2.2) holds with equality, giving indeed “spectral permanence”.

If we specialise to isometric Banach algebra homomorphisms then there is built in a certain degree of spectral permanence, to the extent that we always get

$$\partial \sigma_A(a) \subseteq \sigma_B(Ta);$$

(2.3)

the topological boundary of the larger spectrum is included in the smaller. Equivalently, it turns out, this means that

$$\sigma_A(a) \subseteq \eta \sigma_B(Ta),$$

(2.4)

where the connected hull $\eta K$ of a compact subset $K \subseteq \mathbb{C}$ is the complement of the unbounded connected component of the complement $\mathbb{C} \setminus K$. This has spin-off: if for a particular element $a \in A$ either the larger spectrum is all boundary,

$$\sigma_A(a) \subseteq \partial \sigma_A(a),$$

(2.5)

or the smaller spectrum fills out its connected hull,

$$\eta \sigma_B(Ta) \subseteq \sigma_B(Ta),$$

(2.6)
then the homomorphism \( T : A \to B \) has “spectral permanence at” \( a \in A \), in the sense of equality in (2.2). This holds if for example the spectrum is either real or finite.

If more generally the homomorphism \( T : A \to B \) is one-one there is at least inclusion
\[
is_\sigma A(a) \subseteq \sigma_B(Ta) . \tag{2.7}
\]

3. Generalized permanence

If \( A \) is a semigroup we shall write
\[
A^\cap = \{ a \in A : a \in aAa \} \tag{3.1}
\]
for the “regular” or relatively regular elements of \( A \), those \( a \in A \) which have a generalized inverse \( c \in A \) for which
\[
a = aca : \tag{3.2}
\]
we remark that if (3.2) holds the products
\[
p = ca = p^2 , \quad q = ac = q^2 \tag{3.3}
\]
are both idempotent. Generally if \( T : A \to B \) is a homomorphism there is inclusion
\[
T(A^\cap) \subseteq B^\cap \subseteq B , \tag{3.4}
\]
and hence also
\[
A^\cap \subseteq T^{-1}(B^\cap) \subseteq A . \tag{3.5}
\]
If there is equality in (3.4) we shall say that \( T \) has generalized permanence. This happens for example when
\[
T^{-1}(0) \subseteq A^\cap , \quad T(A) = B : \tag{3.6}
\]
recall the implication
\[
(a - aAa) \cap A^\cap \neq \emptyset \implies a \in A^\cap . \tag{3.7}
\]
This does not however happen when \( T \) is quotienting out the radical as in (1.10), unless the ring \( A \) is semi simple: for notice
\[
\text{Rad}(A) \cap A^\cap = \{0\} . \tag{3.8}
\]

It follows that spectral permanence is not in general sufficient for generalized permanence. Indeed by (3.8) spectral and generalized permanence together imply that a homomorphism \( T : A \to B \) is one one; further (1.5) shows that spectral permanence and one one do not together imply generalized permanence. If \( A \) is the ring of continuous homomorphisms \( a : X \to X \) on a Hausdorff topological
abelian group $X$ then it is necessary for $a \in A \cap \mathbb{A}$ that $a$ have closed range

$$a(X) = \text{cl} \ a(X) :$$  \hspace{1cm} (3.9)

this is because

$$a(X) = ac(X) = (1 - ac)^{-1}(0)$$  \hspace{1cm} (3.10)

is the null space of the complementary idempotent. Thus the embedding (1.6) is another example with spectral but not generalized permanence.

4. Simple permanence

If in particular there is $c \in A$ for which

$$a - aca = 0 = ac - ca ,$$  \hspace{1cm} (4.1)

then $a \in A$ is very special; this happens if $a \in A$ is either invertible, or idempotent, or more generally the commuting product of an invertible and an idempotent. When (4.1) holds we shall say that $a \in A$ is simply polar: in Banach-algebra-land $0 \in \mathbb{C}$ can be at worst a simple pole of the resolvent mapping

$$(z - a)^{-1} : \mathbb{C} \setminus \sigma(a) \to A .$$  \hspace{1cm} (4.2)

In the group theory world the product $cac$ is referred to as the group inverse for $a \in A$. We remark that it is necessary and sufficient for $a \in A$ to be simply polar that

$$a \in a^2 A \cap A a^2 :$$  \hspace{1cm} (4.3)

indeed [15],[19],[20] there is implication

$$a^2u = a = va^2 \implies au = va , \quad aua = a = ava ,$$  \hspace{1cm} (4.4)

giving (4.1) with $c = vau$.

We shall write $\text{SP}(A)$ for the simply polar elements of a semigroup $A$ and observe, for homomorphisms $T : A \to B$, that

$$T \text{ SP}(A) \subseteq \text{SP}(B) \subseteq B ,$$  \hspace{1cm} (4.5)

and hence

$$\text{SP}(A) \subseteq T^{-1} \text{SP}(B) \subseteq A ;$$  \hspace{1cm} (4.6)

when there is equality in (4.5) we shall say that $T : A \to B$ has simple permanence. The counterimage $T^{-1} \text{SP}(B) \subseteq A$ is sometimes known [2],[18],[16] as the “generalized Fredholm” elements of $A$.  

We remark that spectral permanence does not in general, or even
together with one-one-ness, imply simple permanence: return to
(3.8) and (1.5).
In general
\[ \text{SP}(A) \subseteq A^U \equiv \{ a \in A : a \in aA^{-1}a \} , \quad (4.7) \]
and hence
\[ \text{SP}(A) \cap A_{left}^{-1} = A^{-1} = \text{SP}(A) \cap A_{right}^{-1} . \quad (4.8) \]
This will show again that spectral permanence together with one
one is not sufficient for generalized permanence:

**Theorem 4.1.** If \( B_{left}^{-1} \neq B_{right}^{-1} \) then there exist \( A \) and \( T : A \to B \)
for which \( T \) is one one with spectral but not generalized permanence.

**Proof.** If \( A \) is commutative then \( A^{\cap} = \text{SP}(A) \) and hence
\[ T(A^{\cap}) \subseteq \text{SP}(B) \subseteq B^{\cap} , \]
and if
\[ T(A^{\cap}) \cap B_{left}^{-1} \setminus B^{-1} \neq \emptyset \]
then \( T \) does not have generalized permanence. Thus find \( a \in B_{left}^{-1} \setminus B^{-1} \) and, recalling (1.9), take
\[ T = J : \text{comm}_B^2(a) \subseteq B \]

The familiar example is to take \( B = L(X) \) to be the linear mappings
on the space \( X = \mathbb{C}^N \) of all complex sequences and \( a \in B \) to be
the forward shift. Conversely however simple permanence together
with one-one-ness does imply spectral permanence:

**Theorem 4.2.** For semigroup homomorphisms

one one and simple permanence implies spectral permanence , \quad (4.9)

while conversely

simple and spectral permanence implies one one . \quad (4.10)

**Proof.** The last implication is (3.8); conversely observe
\[ \text{SP}(A) \cap T^{-1}B_{left}^{-1} \subseteq A^U \cap T^{-1}B_{left}^{-1} \subseteq A^{-1} + T^{-1}(0) \quad (4.11) \]

\[ \square \]
When we specialise to rings of mappings then simple polarity is characterized by “ascent” and “descent”:

**Theorem 4.3.** If $A = L(X)$ is the additive, or linear, operators on an abelian group, or vector space, $X$ then necessary and sufficient for $a \in A$ to be simply polar is that it has ascent $\leq 1$,

$$a^{-2}(0) \subseteq a^{-1}(0) ; \text{equivalently } a^{-1}(0) \cap a(X) = O \equiv \{0\} , \quad (4.12)$$

and also descent $\leq 1$,

$$a(X) \subseteq a^2(X) ; \text{equivalently } a^{-1}(0) + a(X) = X . \quad (4.13)$$

The same characterization is valid when $A = B(X)$ for a Banach space $X$.

**Proof.** The complementary subspaces $a^{-1}(0)$ and $a(X)$ determine the idempotent $p : X \to X$, defined by setting

$$p(\xi) \in a(X) ; \xi - p(\xi) \in a^{-1}(0)$$

for each $\xi \in X$, whose boundedness, together with the closedness of the range $a(X)$, follows ([7] Theorem 4.8.2) from the open mapping theorem; and finally, if $\xi \in X$,

$$c(\xi) = cp(\xi) ; ca(\xi) = p(\xi)$$

□

We remark that, on incomplete spaces, the conditions (4.5) and (4.6) are not sufficient for simple polarity: indeed it is possible for $a \in B(X)$ to be one one and onto but not in $B(X)^\cap$: the obvious example is the “standard weight” $a = w$ on $X = c_{00} \subseteq c_0$ defined by setting

$$w(\xi)_n = (1/n)\xi_n .$$

Even together with the assumption $a \in A^\cap$, however, the conditions (4.5) and (4.6) are ([7] (7.3.6.8)) not sufficient for simple polarity (4.1) when $A = B(X)$ for an incomplete normed space $X$.

5. **Drazin permanence**

More generally if there is $n \in \mathbb{N}$ for which $a^n$ is simply polar we shall also say that $a \in A$ is “polar”, or Drazin invertible. If $a \in A$ is polar then there is $c \in A$ for which $ac = ca$ and $a - aca$ is nilpotent. If we further relax this to “quasinilpotent” we reach the condition that $a \in A$ “quasipolar”. Specifically if we write

$$QN(A) = \{ a \in A : 1 - Ca \subseteq A^{-1} \} \quad (5.1)$$
for the quasinilpotents of a Banach algebra $A$ then $a \in \text{QN}(A)$ if and only if
\[
\sigma_A(a) \subseteq \{0\},
\]
while with some complex analysis we can prove that if $a \in \text{QN}(A)$ then
\[
\|a^n\|^{1/n} \to 0 \ (n \to \infty).
\] (5.2)
In the ultimate generalization of “group invertibility”, we shall write $\text{QP}(A)$ for the quasipolar elements $a \in A$, those which have a spectral projection $q \in A$ for which (cf [8])
\[
q = q^2; \quad aq = qa; \quad a + q \in A^{-1}; \quad aq \in \text{QN}(A).
\] (5.3)
Now [17] the spectral projection and the Koliha-Drazin inverse
\[
a^* = q, \quad a^\times = (a + q)^{-1}(1 - q)
\] (5.4)
are uniquely determined and lie in the double commutant of $a \in A$. It is easy to see that if (5.3) is satisfied then
\[
0 \notin \text{acc} \quad \sigma_A(a):
\] (5.5)
the origin cannot be an accumulation point of the spectrum; conversely if (5.5) holds then we can display the spectral projection as a sort of “vector-valued winding number”
\[
a^* = \frac{1}{2\pi i} \oint_0 (z - a)^{-1} dz,
\] (5.6)
where we integrate counter clockwise round a small circle $\gamma$ centre the origin whose connected hull $\eta\gamma$ is a disc whose intersection with the spectrum is at most the point $\{0\}$. Now generally for a homomorphism $T : A \to B$ there is inclusion
\[
T \text{QP}(A) \subseteq \text{QP}(B),
\] (5.7)
while if $T : A \to B$ has spectral permanence in the sense (1.3) then it is clear from (5.5) that there is also “Drazin permanence” in the sense that
\[
\text{QP}(A) = T^{-1}\text{QP}(B) \subseteq A:
\] (5.8)
**Theorem 5.1.** For Banach algebra homomorphisms $T : A \to B$ there is implication
\[
\text{spectral permanence } \implies \text{Drazin permanence}.
\]
*Proof.* Equality in (2.2), together with (5.5) □
The example of Theorem 4.1 also shows that the left regular representation \( L : A \rightarrow B(A) \), with \( A = B(X) \) for a normed space \( X \), does not always have generalized permanence; however we do have a sort of “closed range permanence”: there is implication

\[
L_a A = \text{cl } L_a A \implies a(X) = \text{cl } a(X) : \quad (5.9)
\]
indeed if \( a\xi_n \rightarrow \eta \) and \( \varphi \in X^* \) and \( \varphi(\xi) = 1 \) then, with \( \varphi \odot \eta : \zeta \mapsto \varphi(\zeta)\eta \),

\[
L_a(\varphi \odot \eta) = L_a(b) \implies \eta = a(b\xi) . \quad (5.10)
\]

Generally

**Theorem 5.2.** If \( T : A \rightarrow B \) is arbitrary then

\[
\text{QP}(A) \cap T^{-1}(B^{-1}) \subseteq A^{-1} + T^{-1}(0) \quad (5.11)
\]
and if \( T : A \rightarrow B \) is one one then

\[
\text{QP}(A) \cap T^{-1}\text{SP}(B) = \text{SP}(A) . \quad (5.12)
\]

Hence if \( a \in B \) and \( T = J : A = \text{comm}^2_B(a) \subseteq B \) then

\[
A \cap = T^{-1}\text{SP}(B) . \quad (5.13)
\]

It follows that if \( T^{-1}(0) = O \) then

\[
\text{Drazin} \implies \text{simple} \implies \text{spectral permanence} .
\]

*Proof. Uniqueness guarantees that the spectral projection \( T(a)^* \) of \( Ta \in \text{SP}(B) \subseteq \text{QP}(B) \) commutes with \( T(a) \in B \), and one-one-ness guarantees the same for \( a \in A \) \( \square \)*

For Banach algebra homomorphisms therefore there is an improved version of Theorem 4.2 of the three conditions

* spectral permanence ; simple permanence ; one one ,

any two imply the third.

If we rework Theorem 4.1 with \( B = B(\ell_2) \) then it is clear that isometric homomorphisms with spectral permanence need not have generalized permanence: indeed the forward shift \( a = u \in B \cap \text{QP}(A) \text{ is not even quasipolar: we recall that the spectrum of } u \text{ is the closed unit disc, violating (5.5).}

Theorem 4.1 was obtained in this way ([3] Theorem 3.2) in [3]. Of course (cf [9],[17]) “quasinilpotents” and “quasipolars” are only available in Banach algebras; Theorem 4.1 above, using “simply polar” elements, is conceptually much simpler.
6. MOORE-PENROSE PERMANENCE

We recall that a “C* algebra” is a Banach algebra which also has an involution \( a \mapsto a^* \) which is conjugate linear, reverses multiplication, respects the identity and satisfies the “B* condition”

\[
\|a^*a\| = \|a\|^2 \quad (a \in A) .
\]  

(6.1)

Historically the term “C* algebra” was reserved for closed *-subalgebras of the algebras \( B(X) \) for Hilbert spaces \( X \); however the Gelfand-Naimark-Segal (GNS) representation

\[
\Gamma : A \to B(\Xi_A)
\]  

(6.2)

takes an arbitrary “B* algebra” \( A \) isometrically into the algebra of operators on a rather large Hilbert space \( \Xi_A \) built from its “states”: a defect of (6.2) would be that if already \( A = B(X) \) we do not get back \( \Xi_A = X \). In the opinion of this writer these terms “B* algebra” and “C* algebra” could easily ([7] Chapter 8) have been Hilbert algebra. When in particular \( A = B(X) \) for a Hilbert space \( X \) then the closed range condition (3.9) is sufficient for relative regularity \( a \in A^\cap \): indeed we can satisfy (2.2) by setting

\[
c(\xi) = c(q\xi) ; \quad c(a\xi) = p(\xi) \quad (\xi \in X) ,
\]  

(6.3)

where \( q^* = q = q^2 \) and \( p^* = p = p^2 \) are the orthogonal projections on the range \( a(X) \) and the orthogonal complement \( a^{-1}(0)^\perp \) of the null space. The element \( c \in A \) given by (6.3) satisfies four conditions:

\[
a = aca ; \quad c = cac ; \quad (ca)^* = ca ; \quad (ac)^* = ac ,
\]  

(6.4)

and is known as the Moore-Penrose inverse of \( a \in B(X) \): more generally in a C* algebra \( A \) the conditions (6.4) uniquely determine at most one element

\[
c = a^\dagger \in A ,
\]  

(6.5)

lying ([11] Theorem 5) in the double commutant of \( \{a, a^*\} \), and still known as a “Moore-Penrose inverse” for \( a \in A \). Now it is a result of Harte and Mbekhta ([11] Theorem 6) that generally there is equality

\[
A^\cap = A^\dagger ;
\]  

(6.6)

in an arbitrary C* algebra, every relatively regular element has a Moore Penrose inverse. The argument, and a slight generalization, proceeds with the aid of the Drazin inverse.

More generally, on a semigroup \( A \), an involution \( a \mapsto a^* \) satisfies

\[
(a^*)^* = a ; \quad (ca)^* = a^*c^* ; \quad 1^* = 1 .
\]  

(6.7)
In rings and algebras we also ask that the involution be additive, or conjugate linear. The B* condition \((6.7)\) implies that, for arbitrary \(a, x \in A\),
\[
\|ax\|^2 \leq \|x^*\| \|a^*ax\| ,
\]
which in turn gives cancellation
\[
L_{a^*a}(0) \subseteq L_a^{-1}(0) .
\]
Generally the hermitian or “real” elements of \(A\) are given by
\[
\text{Re}(A) = \{ a \in A : a^* = a \} .
\]

The Moore-Penrose inverse \(a^\dagger\) of \((6.4)\), if it exists, is unique and double commutes with \(a\) and \(a^*\). We pause to notice the star polar elements of a semigroup \(A\):
\[
\text{SP}^*(A) = \{ a \in A : a^*a \in A \cap A^\cap \} ;
\]
now we claim

**Theorem 6.1.** If the involution \(*\) on the semigroup \(A\) is cancellable then

\[
A^\dagger \subseteq \text{SP}^*(A) \subseteq A^\cap .
\]

**Proof.** With cancellation there is implication
\[
a \in \text{SP}^*(A) \implies a \in aAa^*a \subseteq A^*a \cap aAa ,
\]
and equality
\[
\text{Re}(A) \cap \text{SP}^*(A) = \text{Re}(A) \cap \text{SP}(A) ,
\]
If \(a = aca\) with \(c = a^\dagger\) then
\[
a^*a = a^*(ac)(ac)^*a = a^*acc^*a^*a \in a^*aAa^*a :
\]
conversely, by cancellation,
\[
a^*a = ada^*a \implies a = ada^*a :
\]
hence also
\[
a \in Aa^*a ; \iff a^* \in a^*aA .
\]

Hence if \(a^* = a\) then \((4.2)\) follows \(\square\)

It is now clear that an isometric C* homomorphism has “Moore-Penrose permanence”:

**Theorem 6.2.** If \(T : A \to B\) has simple permanence then
\[
T^{-1}B^\dagger \subseteq A^\dagger .
\]
Proof. We claim

\[ A^\dagger = \{ a \in A : a^*a \in \text{SP}(A) \} , \]  

with implication

\[ a^*a \in \text{SP}(A) \implies a^\dagger = (a^*a)^*a^* . \]

If \( a \in A^\dagger \) with \( a = aca \) and \( (ca)^* = ca \) then, with \( d = cc^* \), we have

\[ a^*ad = a^*acc^* = a*c = a^*a^*c = ca \]

and

\[ da^*a = cc^*a^*a = ca . \]

Conversely if \( a^*a = a^*ada^*a \) with \( a^*ad = da^*a \) with (wlog) \( d = d^* \) then, using cancellation, with \( c = da^* \),

\[ aca = ada^*a = a \text{ and } ca = da^*a = a^*ad = a^*c* . \]

Now if \( a \in A \) there is implication

\[ Ta \in B^\dagger \implies T(a^*a) \in \text{SP}(B) \implies a^*a \in \text{SP}(A) \implies a \in A^\dagger \]

\[ \square \]

Our main result is a slight generalization, and a new proof, of the Harte/Mbekhta result (6.6), and at the same time “generalized permanence”, equality in (3.4), for isometric C* homomorphisms. One way to go, thanks to the Gelfand/Naimark/Segal representation, is to look first in the very special algebra \( D = B(X) \) of bounded Hilbert space operators:

**Theorem 6.3.** If \( d \in D = B(X) \) for a Hilbert space \( X \) then

\[ (d^*d)^{-1}(0) \subseteq d^{-1}(0) \]  

and

\[ \text{cl } d(X) + d^{*-1}(0) = X ; \]  

hence if \( \text{cl } d(X) = d(X) \) then

\[ d^*(X) = d^*d(X) , \text{ and cl } d^*d(X) = d^*d(X) . \]  

There is inclusion

\[ \text{Re}(D) \cap D^\cap \subseteq \text{SP}(D) ; \]

hence

\[ d \in D^\cap \implies d \in \text{SP}^*(D) \implies d^*d \in \text{SP}(D) \implies d \in D^\dagger . \]
Proof. For arbitrary $\xi \in X$ there is [3] inequality
$$\|d\xi\|^2 \leq \|\xi\| \|d^*d\xi\|,$$
and also
$$\text{cl } d(X) = d^{*-1}(0)^\perp$$
\[
\square
\]
Both of the Harte/Mbekhta observations now follow:

**Theorem 6.4.** If $T : A \to B$ is isometric then
$$T^{-1}(B^\cap) \subseteq A^\dagger.$$  \hfill (6.20)

**Proof.** With $S : B \to D = B(X)$ a GNS mapping we argue, using again Theorem 4.2, together with “spectral permanence at” $a^*a$ (which has of course real spectrum),
$$Ta \in B^\cap \implies ST(a^*a) \in \text{SP}(D) \implies a^*a \in \text{SP}(A) \implies a \in A^\dagger$$
\[
\square
\]
In the situation of (6.14),
$$a = a^* \in A^\cap \implies a^\dagger = a^\times ; 1 - a^\dagger a = a^\bullet.$$  \hfill (6.21)

Theorem [6.4] has an obvious extension to homomorphisms with closed range:

**Theorem 6.5.** If $T : A \to B$ has closed range then there is implication, for arbitrary $a \in A$,
$$T(a) \in B^\cap \implies a + T^{-1}(0) \in (A/T^{-1}(0))^\cap.$$  \hfill (6.22)

**Proof.** Apply Theorem [6.4] to the bounded below $T^\wedge : A/T^{-1}(0) \to B$ \[
\square
\]

**References**

Robin Harte taught for twenty years at University College, Cork, and retired a long time ago, but somehow never goes away: his ghost can sometimes be seen at the TCD-UCD Analysis Seminar.

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