Cartan Subalgebras in C*-Algebras

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Abstract. According to J. Feldman and C. Moore's well-known theorem on Cartan subalgebras, a variant of the group measure space construction gives an equivalence of categories between twisted countable standard measured equivalence relations and Cartan pairs, i.e., a von Neumann algebra (on a separable Hilbert space) together with a Cartan subalgebra. A. Kumjian gave a C*-algebraic analogue of this theorem in the early eighties. After a short survey of maximal abelian self-adjoint subalgebras in operator algebras, I present a natural definition of a Cartan subalgebra in a C*-algebra and an extension of Kumjian's theorem which covers graph algebras and some foliation algebras.

1. Introduction

One of the most fundamental constructions in the theory of operator algebras, namely the crossed product construction, provides a subalgebra, i.e., a pair (B, A) consisting of an operator algebra A and a subalgebra $B \subset A$, where B is the original algebra. The inclusion $B \subset A$ encodes the symmetries of the original dynamical system. An obvious and naive question is to ask whether a given subalgebra arises from some crossed product construction. From the very construction of the crossed product, a necessary condition is that B is regular in A, which means that A is generated by the normalizer of B. In the case of a crossed product by a group, duality theory provides an answer (see Landstad [30]) which requires an external information, namely the dual action. Our question is more in line with subfactor theory, where one extracts an algebraic object (such as a paragroup or a quantum groupoid) solely from an inclusion

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of factors. Under the assumption that B is maximal abelian, the problem is somewhat more tractable. The most satisfactory result in this direction is the Feldman–Moore theorem [18, Theorem 1], which characterizes the subalgebras arising from the construction of the von Neumann algebra of a measured countable equivalence relation. These subalgebras are precisely the Cartan subalgebras, a nice kind of maximal abelian self-adjoint subalgebras (masas) introduced previously by Vershik in [47]: they are regular and there exists a faithful normal conditional expectation of A onto B. The Cartan subalgebra contains exactly the same information as the equivalence relation. This theorem leaves pending a number of interesting and difficult questions. For example, the existence or the uniqueness of Cartan subalgebras in a given von Neumann algebra. Another question is to determine if the equivalence relation arises from a free action of a countable group and if one can expect uniqueness of the group. There have been some recent breakthroughs on these questions: for example [36, 37, 38, 33]; in [33], Ozawa and Popa give the first examples of II_1 factors containing a unique Cartan subalgebra up to unitary conjugacy.

It was then natural to find a counterpart of the Feldman–Moore theorem for C*-algebras. In [25], Kumjian introduced the notion of a C*-diagonal as the C*-algebraic counterpart of a Cartan subalgebra and showed that, via the groupoid algebra construction, they correspond exactly to twisted étale equivalence relations. A key ingredient of his theorem is his definition of the normalizer of a subalgebra (a definition in terms of unitaries or partial isometries would be too restrictive). His fundamental result, however, does not cover a number of important examples. For example, Cuntz algebras, and more generally graph algebras, have obvious regular masas which are not C*-diagonals. The same is true for foliations algebras (or rather their reduction to a full transversal). The reason is that the groupoids from which they are constructed are topologically principal but not principal: they have some isotropy that cannot be eliminated. It seems that, in the topological context, topologically principal groupoids are more natural than principal groupoids (equivalence relations). They are exactly the groupoids of germs of pseudogroups. Groupoids of germs of pseudogroups present a technical difficulty: they may fail to be Hausdorff (they are Hausdorff if and only if the pseudogroup is quasi-analytical). For the sake of simplicity, our discussion will be limited to the Hausdorff case. We refer the interested reader to a forthcoming paper about the non-Hausdorff case. A natural definition of a Cartan subalgebra in the C*-algebraic context is that it is a masa which is regular and which admits a faithful conditional expectation. We show that in the reduced C*-algebra of a topologically principal Hausdorff étale groupoid (endowed with a twist), the subalgebra corresponding to the unit space is a Cartan subalgebra. Conversely, every Cartan subalgebra (if it exists!) arises in that fashion and completely determines the groupoid and the twist. Our proof closely follows Kumjian's. The comparison with Kumjian's theorem shows that a Cartan subalgebra has the unique extension property if and only if the corresponding groupoid is principal. As a corollary of the main result, we obtain that a Cartan subalgebra has a unique conditional expectation, which is clear when the subalgebra has the unique extension property but not so in the general case.

Here is a brief description of the content of this paper. In Section 2, I will review some basic facts about masas in von Neumann algebras, the Feldman–Moore theorem and some more recent results on Cartan subalgebras. In Section 3, I will review the characterization of topologically principal groupoids as groupoids of germs of pseudogroups of local homeomorphisms. In Section 4, I will review the construction of the reduced C*-algebra of a locally compact Hausdorff groupoid G with Haar system and endowed with a twist. I will show that, when G is étale, the subalgebra of the unit space is a masa if and only if G is topologically principal. In fact, this is what we call a Cartan subalgebra in the C*-algebraic context: it means a masa which is regular and which has a faithful conditional expectation. In Section 5, we show the converse: every Cartan subalgebra arises from an topologically principal étale groupoid endowed with a twist. This groupoid together with its twist is a complete isomorphism invariant of the Cartan subalgebra. We end with examples of Cartan subalgebras in C*-algebras.

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2. Cartan Subalgebras in von Neumann Algebras

The basic example of a masa in an operator algebra is the subalgebra D_n of diagonal matrices in the algebra M_n of complex-valued (n, n)-matrices. Every masa in M_n is conjugated to it by a unitary (this is essentially the well-known result that every normal complex matrix admits an orthonormal basis of eigenvectors). The problem at hand is to find suitable generalizations of this basic example.

The most immediate generalization is to replace $\mathbf{C}^n$ by an infinite dimensional separable Hilbert space H and M_n by the von Neumann algebra $\mathcal{B}(H)$ of all bounded linear operators on H. The spectral theorem tells us that, up to conjugation by a unitary, masas in $\mathcal{B}(H)$ are of the form $L^\infty(X)$, acting by multiplication on $H = L^2(X)$, where X is an infinite standard measure space. Usually, one distinguishes the case of $X = [0, 1]$ endowed with Lebesgue measure and the case of $X = \mathbf{N}$ endowed with counting measure. In the first case, the masa is called *diffuse* and in the second case, it is called *atomic*. Atomic masas A in $\mathcal{B}(H)$ can be characterized by the existence of a normal conditional expectation $P : \mathcal{B}(H) \to A$. Indeed, when $H = \ell^2(\mathbf{N})$, operators are given by matrices and P is the restriction to the diagonal. What we are looking for is precisely a generalization of these atomic masas.

There is no complete classification of masas in non-type I factors. In fact, the study of masas in non-type I factors looks like a rather formidable task. In 1954, J. Dixmier [13] discovered the existence of non-regular masas. A masa A in a von Neumann algebra M is called *regular* if its normalizer $N(A)$ (the group of unitaries u in M which normalize A, in the sense that $uAu^* = A$) generates M as a von Neumann algebra. On the other hand, it is called *singular* if $N(A)$ is contained in A. When $N(A)$ acts ergodically on A, the masa A is called *semi-regular*. Every masa in $\mathcal{B}(H)$ (or in a type I von Neumann algebra) is regular. Dixmier gave an example of a singular masa in the hyperfinite II_1 factor. Later, Popa has shown in [35] that singular masas do exist in every separable II_1 factor (as we shall see, this is in sharp contrast with regular masas). Moreover, every von Neumann subalgebra of a separable II_1 factor is the image of a normal conditional expectation. Thus, in order to generalize the atomic masas of $\mathcal{B}(H)$, it is natural to consider masas which are both regular and the image of a normal conditional expectation:

Definition 2.1. (Vershik [47], Feldman–Moore [18, Definition 3.1]) An abelian subalgebra A of a von Neumann algebra M is called a *Cartan subalgebra* if

(i) A is a masa;
(ii) A is regular;
(iii) there exists a faithful normal conditional expectation of M onto A.

Cartan subalgebras are intimately related to ergodic theory. Indeed, if M arises by the classical group measure construction from a free action of a discrete countable group Γ on a measure space (X, μ), then $L^\infty(X, \mu)$ is naturally imbedded in M as a Cartan subalgebra ([32]). Following generalizations by G. Zeller-Meier [50, Remarque 8.11], W. Krieger [21] and P. Hahn [20], J. Feldman and C. Moore give in [18] the most direct construction of Cartan subalgebras. It relies on the notion of a countable standard measured equivalence relation. Here is its definition: $(X, \mathcal{B}, \mu)$ is a standard measured space and R is an equivalence relation on X such that its classes are countable, its graph R is a Borel subset of $X \times X$ and the measure μ is quasi-invariant under R. The last condition means that the measures $r^*\mu$ and $s^*\mu$ on R are equivalent (where r, s denote respectively the first and the second projections of R onto X and $r^*\mu(f) = \int \sum_y f(x, y) d\mu(x)$ for a positive Borel function f on R). The orbit equivalence relation of an action of a discrete countable group Γ on a measure space (X, μ) preserving the measure class of μ is an example (in fact, according to [17, Theorem 1], it is the most general example) of a countable standard equivalence relation. The construction of the von Neumann algebra $M = W^*(R)$ mimicks the construction of the algebra of matrices M_n. Its elements are complex Borel functions on R, the product is matrix multiplication and involution is the usual matrix conjugation. Of course, in order to have an involutive algebra of bounded operators, some conditions are required on these functions: they act by left multiplication as operators on $L^2(R, s^*\mu)$ and we ask these operators to be bounded. The subalgebra A of diagonal matrices (functions supported on the diagonal of R), which is isomorphic to $L^\infty(X, \mu)$, is a Cartan subalgebra of M. When $X = \mathbf{N}$ and μ is the counting measure, one retrieves the atomic masa of $\mathcal{B}(\ell^2(\mathbf{N}))$. This construction can be twisted by a 2-cocycle $\sigma \in Z^2(R, \mathbf{T})$; explicitly, σ is a Borel function on $R^{(2)} = \{(x, y, z) \in X \times X \times X : (x, y), (y, z) \in R\}$

with values in the group of complex numbers of modulus 1 such that $\sigma(x,y,z)\sigma(x,z,t) = \sigma(x,y,t)\sigma(y,z,t)$. The only modification is to define as product the twisted matrix multiplication $f * g(x,z) = \sum f(x,y)g(y,z)\sigma(x,y,z)$. This yields the von Neumann algebra $M = W^*(R,\sigma)$ and its Cartan subalgebra $A = L^\infty(X,\mu)$ of diagonal matrices. The Feldman–Moore theorem gives the converse.

Theorem 2.2. [18, Theorem 1] *Let A be a Cartan subalgebra of a von Neumann algebra M on a separable Hilbert space. Then there exists a countable standard measured equivalence relation R on (X,μ), a $\sigma \in Z^2(R,\mathbf{T})$ and an isomorphism of M onto $W^*(R,\sigma)$ carrying A onto the diagonal subalgebra $L^\infty(X,\mu)$. The twisted relation (R,σ) is unique up to isomorphism.*

The main lines of the proof will be found in the C*-algebraic version of this result. This theorem completely elucidates the structure of Cartan subalgebras. It says nothing about the existence and the uniqueness of Cartan subalgebras in a given von Neumann algebra. We have seen that in $\mathcal{B}(H)$ itself, there exists a Cartan subalgebra, which is unique up to conjugacy. The same result holds in every injective von Neumann algebra. More precisely, two Cartan subalgebras of an injective von Neumann algebra are always conjugate by an automorphism (but not always inner conjugate, as observed in [18]). This important uniqueness result appears as [7, Corollary 11]. W. Krieger had previously shown in [22, Theorem 8.4] that two Cartan subalgebras of a von Neumann algebra M which produce hyperfinite ([17, Definition 4.1]) equivalence relations are conjugate (then, M is necessarily hyperfinite). On the other hand, it is not difficult to show that a Cartan subalgebra of an injective von Neumann algebra produces an amenable ([7, Definition 6]) equivalence relation. Since Connes–Feldman–Weiss's theorem states that an equivalence relation is amenable if and only if it is hyperfinite, Krieger's uniqueness theorem can be applied. The general situation is more complex. Here are some results related to Cartan subalgebras of type II_1 factors. In [8], A. Connes and V. Jones give an example of a II_1 factor with at least two non-conjugate Cartan subalgebras. Then S. Popa constructs in [36] a II_1 factor with uncountably many non-conjugate Cartan subalgebras. These examples use Kazhdan's property T. In [48], D. Voiculescu shows that for $n \geq 2$, the von Neumann algebra $L(\mathbf{F}_n)$ of the free group $\mathbf{F}_n$ on n generators has no Cartan subalgebra. Despite these rather negative results, it seems that the notion of Cartan subalgebra still has a rôle to play in the theory of II_1 factors. For example, S. Popa has recently (see [37, 38]) constructed and studied a large class of type II_1 factors (from Bernoulli actions of groups with property T) which have a distinguished Cartan subalgebra, unique up to inner conjugacy; moreover, these factors satisfy remarkable rigidity properties: isomorphisms of these factors essentially arise from conjugacy of the actions. Still more recently N. Ozawa and S. Popa give in [33] on one hand many examples of II_1 factors which do not have any Cartan subalgebra and on the other hand a new class of II_1 factors which have a unique Cartan subalgebra, in fact unique not only up to conjugacy but to inner conjugacy. This class consists of all the profinite ergodic probability preserving actions of free groups $\mathbf{F}_n$ with $n \geq 2$ and their products.

3. Topologically Principal Groupoids.

The purpose of this section is mainly notational. It recalls elementary facts about étale groupoids and pseudogroups of homeomorphisms. Concerning groupoids, we shall use the notation of [1]. Other relevant references are [40] and [34]. Given a groupoid G, $G^{(0)}$ will denote its unit space and $G^{(2)}$ the set of composable pairs. Usually, elements of G will be denoted by Greek letters as γ and elements of $G^{(0)}$ by Roman letters as x, y. The range and source maps from G to $G^{(0)}$ will be denoted respectively by r and s. The fibers of the range and source maps are denoted respectively $G^x = r^{-1}(x)$ and $G_y = s^{-1}(y)$. The inverse map $G \to G$ is written $\gamma \mapsto \gamma^{-1}$, the inclusion map $G^{(0)} \to G$ is written $x \mapsto x$ and the product map $G^{(2)} \to G$ is written $(\gamma, \gamma') \mapsto \gamma\gamma'$. The *isotropy bundle* is $G' = \{\gamma \in G : r(\gamma) = s(\gamma)\}$. It is the disjoint union of the isotropy subgroups $G(x) = G^x \cap G_x$ when x runs over $G^{(0)}$.

In the topological setting, we assume that the groupoid G is a topological space and that the structure maps are continuous, where $G^{(2)}$ has the topology induced by $G \times G$ and $G^{(0)}$ has the topology induced by G. We assume furthermore that the range and source maps are open. A topological groupoid G is called *étale* when its range and source maps are local homeomorphisms from G onto $G^{(0)}$.

We shall be exclusively concerned here with groupoids of germs. They are intimately connected with pseudogroups. Here are the definitions. Let X be a topological space. A homeomorphism $\varphi : U \to V$, where U, V are open subsets of X, is called a partial homeomorphism.

Under composition and inverse, the partial homeomorphisms of X form an inverse semigroup. A *pseudogroup* on X is a family $\mathcal{G}$ of partial homeomorphisms of X stable under composition and inverse. We say that the pseudogroup $\mathcal{G}$ is *ample* if every partial homeomorphism φ which locally belongs to $\mathcal{G}$ (i.e., every point in the domain of φ has an open neighborhood U such that $\varphi_{|U} = \beta_{|U}$ with $\beta \in \mathcal{G}$) does belong to $\mathcal{G}$. Given a pseudogroup $\mathcal{G}$, we denote by $[\mathcal{G}]$ the set of partial homeomorphisms which belong locally to $\mathcal{G}$; it is an ample pseudogroup called the ample pseudogroup of $\mathcal{G}$. Given a pseudogroup $\mathcal{G}$ on the topological space X, its *groupoid of germs* is

$$G = \{[x, \varphi, y], \quad \varphi \in \mathcal{G}, y \in dom(\varphi), x = \varphi(y)\}.$$

where $[x, \varphi, y] = [x, \psi, y]$ if and only if φ and ψ have the same germ at y, i.e., there exists a neighborhood V of y in X such that $\varphi_{|V} = \psi_{|V}$. Its groupoid structure is defined by the range and source maps $r[x, \varphi, y] = x, s[x, \varphi, y] = y$, the product $[x, \varphi, y][y, \psi, z] = [x, \varphi\psi, z]$ and the inverse $[x, \varphi, y]^{-1} = [y, \varphi^{-1}, x]$. Its topology is the topology of germs, defined by the basic open sets

$$\mathcal{U}(U, \varphi, V) = \{[x, \varphi, y] \in G : x \in U, y \in V\}$$

where U, V are open subsets of X and $\varphi \in \mathcal{G}$. Observe that the groupoid of germs G of the pseudogroup $\mathcal{G}$ on X depends on the ample pseudogroup $[\mathcal{G}]$ only.

Conversely, an étale groupoid G defines a pseudogroup $\mathcal{G}$ on $X = G^{(0)}$ as follows. Recall that a subset A of a groupoid G is called an *r-section* [resp. an *s-section*] if the restriction of r [resp. s] to A is injective. A *bisection* is a subset $S \subset G$ which is both an r-section and an s-section. If G is an étale topological groupoid, it has a cover of open bisections. The open bisections of an étale groupoid G form an *inverse semigroup* $\mathcal{S} = \mathcal{S}(G)$: the composition law is

$$ST = \{\gamma\gamma' : (\gamma, \gamma') \in (S \times T) \cap G^{(2)}\}$$

and the inverse of S is the image of S by the inverse map. The inverse semigroup relations, which are $(RS)T = R(ST)$, $(ST)^{-1} = T^{-1}S^{-1}$ and $SS^{-1}S = S$, are indeed satisfied. A bisection S defines a map $\alpha_S : s(S) \to r(S)$ such that $\alpha_S(x) = r(Sx)$ for $x \in s(S)$. If moreover G is étale and S is an open bisection, this map is a homeomorphism. The map $\alpha : S \mapsto \alpha_S$ is an inverse semigroup homomorphism of the inverse semigroup of open bisections $\mathcal{S}$ into the inverse semigroup of partial homeomorphisms of X. We call it the *canonical action* of $\mathcal{S}$ on X. The relevant pseudogroup is its range $\mathcal{G} = \alpha(\mathcal{S})$.

Proposition 3.1. *Let $\mathcal{G}$ be a pseudogroup on X, let G be its groupoid of germs and let $\mathcal{S}$ be the inverse semigroup of open bisections of G. Then*

(i) *The pseudogroup $\alpha(\mathcal{S})$ is the ample pseudogroup $[\mathcal{G}]$ of $\mathcal{G}$.*
(ii) *The canonical action α is an isomorphism from $\mathcal{S}$ onto $[\mathcal{G}]$.*

Proof. We have observed above that $\mathcal{G}$ and $[\mathcal{G}]$ define the same groupoid of germs G. Thus, every $\varphi \in [\mathcal{G}]$ defines the open bisection $S = S_\varphi = \mathcal{U}(X, \varphi, X)$. By construction, $\alpha_S = \varphi$. Conversely, let S be an open bisection of G. It can be written as a union $S = \cup_i \mathcal{U}(V_i, \varphi_i, U_i)$, where U_i, V_i are open subsets of X and $\varphi_i \in \mathcal{G}$. This shows that $\varphi = \alpha_S$ belongs to $[\mathcal{G}]$ and that $S_\varphi = S$. In other words, the maps $S \to \alpha_S$ and $\varphi \to S_\varphi$ are inverse of each other. □

Proposition 3.2. *Let G be an étale groupoid over X and let α be the canonical action of the inverse semigroup of its open bisections $\mathcal{S}$ on X. Let H be the groupoid of germs of the pseudogroup $\alpha(\mathcal{S})$. Then we have a short exact sequence of étale groupoids*

$$\mathrm{int}(G') \rightarrowtail G \twoheadrightarrow H$$

where $\mathrm{int}(G')$ *is the interior of the isotropy bundle.*

Proof. We define $\alpha_* : G \to H$ by sending $\gamma \in G$ into $[r(\gamma), \alpha_S, s(\gamma)]$, where S is an open bisection containing γ. This does not depend on the choice of S, because α_S, α_T and $\alpha_{S\cap T}$, where T is another open bisection containing γ, have the same germ at $s(\gamma)$. It is readily verified that α_* is a continuous and surjective homomorphism. Moreover, $\alpha_*(\gamma)$ is a unit in H if and only if the germ of S at $s(\gamma)$ is the identity. This happens if and only if γ belongs to the interior of G' because α_S is an identity map if and only if S is contained in G'. □

Corollary 3.3. *Let G be an étale groupoid over X and let $\mathcal{S}$ be the inverse semigroup of its open bisections. Let α be the canonical action of $\mathcal{S}$ on X. The following properties are equivalent:*

(i) *The map α is one-to-one.*
(ii) *The interior of G' is reduced to $G^{(0)}$.*

Proof. Assume that the map α is one-to-one. Then, the above map $\alpha_* : G \to H$ is one-to-one. Hence its kernel $\mathrm{int}(G')$ is reduced to $G^{(0)}$.

Conversely, if $\mathrm{int}(G') = G^{(0)}$, then G is isomorphic to H. Hence it is a groupoid of germs. Therefore, according to Proposition 3.1(ii), α is one-to-one. □

Definition 3.4. An étale groupoid which satisfies above equivalent conditions is called *effective*.

The reader will find a good discussion of this notion in the monograph [31, Section 5.5] by I. Moerdijk and J. Mrčun .

Definition 3.5. Let us a say that an étale groupoid G is

(i) *principal* if $G' = G^{(0)}$

(ii) *topologically principal* if the set of points of $G^{(0)}$ with trivial isotropy is dense.

The property *(ii)* appears under the name *essentially principal* in some previous articles (e.g., [3]). The present terminology agrees with the notion of a topologically free action introduced in [45, Definition 2.1].

The following proposition links effective groupoids and topologically principal groupoids.

Proposition 3.6. *Let G be an étale groupoid.*

(i) *If G is Hausdorff and topologically principal, then it is effective;*

(ii) *If G is a second countable effective groupoid and its unit space $G^{(0)}$ has the Baire property, then it is topologically principal.*

Proof. Let us introduce the set Y of units with trivial isotropy and its complement $Z = G^{(0)} \setminus Y$. Let us suppose that G is topologically principal. Then Z has an empty interior. Let U be an open subset of G contained in G'. Since G is Hausdorff, $G^{(0)}$ is closed in G and $U \setminus G^{(0)}$ is open. Therefore $r(U \setminus G^{(0)})$, which is open and contained in Z, is empty. This implies that $U \setminus G^{(0)}$ itself is empty and that U is contained in $G^{(0)}$.

Let us assume that G is second countable, that its unit space $G^{(0)}$ has the Baire property and that it is effective. We choose a countable family (S_n) of open bisections which covers G. We introduce the subsets $A_n = r(S_n \cap G')$ of $G^{(0)}$. By definition, for each n, $Y_n = int(A_n) \cup ext(A_n)$ is a dense open subset of $G^{(0)}$. By the Baire property, the intersection $\cap_n Y_n$ is dense in $G^{(0)}$. Let us show that

$\cap_n Y_n$ is contained in Y. Suppose that x belongs to $\cap_n Y_n$ and that γ belongs to $G(x)$. There exists n such that γ belongs to S_n. Then γ belongs to $S_n \cap G'$ and $x = r(\gamma)$ belongs to A_n. Since it also belongs to Y_n, it must belong to $int(A_n)$. Let V be an open set containing x and contained in A_n. Since r is a bijection from $S_n \cap G'$ onto A_n, the open set VS_n is contained in G'. According to condition (ii) of Corollary 3.3, it is contained in $G^{(0)}$ and $\gamma = xS_n$ belongs to $G^{(0)}$. Therefore x belongs to Y. $\square$

There are easy examples of groupoids of germs which are not topologically principal. For example, the groupoid of germs of the pseudogroup of all partial homeomorphisms of $\mathbf{R}$, which is transitive, is not topologically principal.

4. The Analysis of the Twisted Groupoid C*-Algebra.

Following [25], one defines a twisted groupoid as a central groupoid extension

$$\mathbf{T} \times G^{(0)} \rightarrowtail \Sigma \twoheadrightarrow G$$

where $\mathbf{T}$ is the circle group. Thus, Σ is a groupoid containing $\mathbf{T} \times G^{(0)}$ as a subgroupoid. One says that Σ is a twist over G. We assume that Σ and G are topological groupoids. In particular, Σ is a principal $\mathbf{T}$-space and $\Sigma/\mathbf{T} = G$. We form the associated complex line bundle $L = (\mathbf{C} \times \Sigma)/\mathbf{T}$ over G, where $\mathbf{T}$ acts by the diagonal action $z(\lambda, \sigma) = (\lambda\overline{z}, z\sigma)$. The class of (λ, σ) is written $[\lambda, \sigma]$. We write $\dot\sigma \in G$ the image of $\sigma \in \Sigma$. The line bundle L is a Fell bundle over the groupoid G, as defined in [27] (see also [16]): it has the product $L_{\dot\sigma} \otimes L_{\dot\tau} \to L_{\dot\sigma\dot\tau}$, sending $([\lambda, \sigma], [\mu, \tau])$ into $[\lambda\mu, \sigma\tau]$ and the involution $L_{\dot\sigma} \to L_{\dot\sigma^{-1}}$ sending $[\lambda, \sigma]$ into $[\overline{\lambda}, \sigma^{-1}]$. An element u of a Fell bundle L is called unitary if u^*u and uu^* are unit elements. The set of unitary elements of L can be identified to Σ through the map $\sigma \in \Sigma \mapsto [1, \sigma] \in L$. In fact, this gives a one-to-one correspondence between twists over G and Fell line bundles over G (see [12]). It is convenient to view the sections of L as complex-valued functions $f : \Sigma \to \mathbf{C}$ satisfying $f(z\sigma) = f(\sigma)\overline{z}$ for all $z \in \mathbf{T}, \sigma \in \Sigma$ and we shall usually do so. When there is no risk of confusion, we shall use the same symbol for the function f and the section of L it defines.

In order to define the twisted convolution algebra, we assume from now on that G is locally compact, Hausdorff, second countable and that it possesses a Haar system λ. It is a family of measures $\{\lambda_x\}$ on G, indexed by $x \in G^{(0)}$, such that λ_x has exactly G^x as its support, which is continuous, in the sense that for every $f \in C_c(G)$, the function $\lambda(f) : x \mapsto \lambda_x(f)$ is continuous, and invariant, in the sense that for every $\gamma \in G$, $R(\gamma)\lambda_{r(\gamma)} = \lambda_{s(\gamma)}$, where $R(\gamma)\gamma' = \gamma'\gamma$. When G is an étale groupoid, it has a canonical Haar system, namely the counting measures on the fibers of s.

Let (G, λ) be a Hausdorff locally compact second countable groupoid with Haar system and let Σ be a twist over G. We denote by $C_c(G, \Sigma)$ the space of continuous sections with compact support of the line bundle associated with Σ. The following operations

$$f * g(\sigma) = \int f(\sigma\tau^{-1})g(\tau)d\lambda_{s(\sigma)}(\dot\tau) \tag{1}$$

$$f^*(\sigma) = \overline{f(\sigma^{-1})} \tag{2}$$

turn $C_c(G, \Sigma)$ into a $*$-algebra. Furthermore, we define for $x \in G^{(0)}$ the Hilbert space $H_x = L^2(G_x, L_x, \lambda_x)$ of square-integrable sections of the line bundle $L_x = L_{|G_x}$. Then, for $f \in C_c(G, \Sigma)$, the operator $\pi_x(f)$ on H_x defined by

$$\pi_x(f)\xi(\sigma) = \int f(\sigma\tau^{-1})\xi(\tau)d\lambda_x(\dot\tau)$$

is bounded. This can be deduced from the useful estimate:

$$\|\pi_x(f)\| \le \|f\|_I = \max\left(\sup_y \int |f| d\lambda_y, \sup_y \int |f^*| d\lambda_y\right).$$

Moreover, the field $x \mapsto \pi_x(f)$ is continuous when the family of Hilbert spaces H_x is given the structure of a continuous field of Hilbert spaces by choosing $C_c(G, \Sigma)$ as a fundamental family of continuous sections. Equivalently, the space of sections $C_0(G^{(0)}, H)$ is a right C*-module over $C_0(G^{(0)})$ and π is a representation of $C_c(G, \Sigma)$ on this C*-module.

The reduced C*-algebra $C^*_{red}(G, \Sigma)$ is the completion of $C_c(G, \Sigma)$ with respect to the norm $\|f\| = \sup_x \|\pi_x(f)\|$.

Let us now study the properties of the pair $(A = C^*_{red}(G, \Sigma), B = C_0(G^{(0)}))$ that we have constructed from a twisted étale Hausdorff locally compact second countable groupoid (G, Σ).

The main technical tool is that the elements of the reduced C*-algebra $C^*_{red}(G, \Sigma)$ are still functions on Σ (or sections of the line bundle L).

Proposition 4.1. [40, II.4.1] *Let G be an étale Hausdorff locally compact second countable groupoid and let Σ be a twist over G. Then, for all $f \in C_c(G,\Sigma)$ we have:*

(i) $|f(\sigma)| \leq \|f\|$ *for every* $\sigma \in \Sigma$ *and*
(ii) $\int |f|^2 d\lambda_x \leq \|f\|^2$ *for every* $x \in G^{(0)}$.

Proof. This is easily deduced (see [40, II.4.1]) from the following equalities:

$$f(\sigma) = < \epsilon_\sigma, \pi_{s(\sigma)}(f)\epsilon_{s(\sigma)} >, \quad f_{|\Sigma_x} = \pi_x(f)\epsilon_x,$$

where $f \in C_c(G,\Sigma)$, $\sigma \in \Sigma$, $x \in G^{(0)}$ and $\epsilon_\sigma \in H_{s(\sigma)}$ is defined by $\epsilon_\sigma(\tau) = \overline{z}$ if $\tau = z\sigma$ and 0 otherwise. □

As a consequence ([40, II.4.2]) the elements of $C^*_{red}(G,\Sigma)$ can be viewed as continuous sections of the line bundle L. Moreover, the equations (1) and (2) defining $f * g$ and f^* are still valid for $f, g \in C^*_{red}(G,\Sigma)$ (the sum defining $f * g(\sigma)$ is convergent). It will be convenient to define the open support of a continuous section f of the line bundle L as

$$supp'(f) = \{\gamma \in G : f(\gamma) \neq 0\}.$$

Note that the unit space $G^{(0)}$ of G is an open (and closed) subset of G and that the restrictions of the twist Σ and of the line bundle L to $G^{(0)}$ are trivial. We have the following identification:

$$C_0(G^{(0)}) = \{f \in C^*_{red}(G,\Sigma) : supp'(f) \subset G^{(0)}\}$$

where $h \in C_0(G^{(0)})$ defines the section f defined by $f(\sigma) = h(x)\overline{z}$ if $\sigma = (x,z)$ belongs to $G^{(0)} \times \mathbf{T}$ and $f(\sigma) = 0$ otherwise. Then $B = C_0(G^{(0)})$ is an abelian sub-C*-algebra of $A = C^*_{red}(G,\Sigma)$ which contains an approximate unit of A.

Here is an important application of the fact that the elements of $C^*_{red}(G,\Sigma)$ can be viewed as continuous sections.

Theorem 4.2. [40, II.4.7] *Let (G,Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $A = C^*_{red}(G,\Sigma)$ and $B = C_0(G^{(0)})$. Then*

(i) *an element $a \in A$ commutes with every element of B if and only if its open support $supp'(a)$ is contained in G';*
(ii) *B is a masa if and only if G is topologically principal.*

Proof. Since the elements of $C^*_{red}(G,\Sigma)$ are continuous sections of the associated line bundle L, it is straightforward to spell out the condition $ab = ba$ for all $b \in B$. It implies the given condition on the support of a. We refer to [40, II.4.7] for details. One deduces from (i) that B is a masa if and only if the interior of G' is $G^{(0)}$. According to Proposition 3.6, this is equivalent under our hypotheses to G being topologically principal. □

Another piece of structure of the pair $(A = C^*_{red}(G,\Sigma), B = C_0(G^{(0)}))$ is the restriction map $P : f \mapsto f_{|G^{(0)}}$ from A to B.

Proposition 4.3. [40, II.4.8] *Let (G,Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $P : C^*_{red}(G,\Sigma) \to C_0(G^{(0)})$ be the restriction map. Then*

(i) *P is a conditional expectation onto $C_0(G^{(0)})$.*
(ii) *P is faithful.*
(iii) *If G is topologically principal, P is the unique conditional expectation onto $C_0(G^{(0)})$.*

Proof. This is proved in [40, II.4.8] in the principal case. The main point of (i) is that P is well defined, which is clear from the above. There is no difficulty checking that it has all the properties of an expectation map. Note that for $h \in C_0(G^{(0)})$ and $f \in C^*_{red}(G,\Sigma)$, we have $(hf)(\sigma) = h(r(\sigma))f(\sigma)$ and $(fh)(\sigma) = f(\sigma)h(s(\sigma))$. The assertion (ii) is also clear: for $f \in C^*_{red}(G,\Sigma)$ and $x \in G^{(0)}$, we have

$$P(f^* * f)(x) = \int |f(\tau)|^2 d\lambda_x(\dot{\tau}).$$

Hence, if $P(f^* * f) = 0$, $f(\tau) = 0$ for all $\tau \in \Sigma$. Let us prove (iii). Let $Q : C^*_{red}(G,\Sigma) \to C_0(G^{(0)})$ be a conditional expectation. We shall show that Q and P agree on $C_c(G,\Sigma)$, which suffices to prove the assertion. Let $f \in C_c(G,\Sigma)$ with compact support K in G. We first consider the case when K is contained in an open bisection S which does not meet $G^{(0)}$ and show that $Q(f) = 0$. If $x \in G^{(0)}$ does not belong to $s(K)$, then $Q(f)(x) = 0$. Indeed, we choose $h \in C_c(G^{(0)})$ such that $h(x) = 1$ and its support does not meet $s(K)$. Then $fh = 0$, therefore $Q(f)(x) = Q(f)(x)h(x) = (Q(f)h)(x) = Q(fh)(x) = 0$. Let $x_0 \in G^{(0)}$ be such that $Q(f)(x_0) \neq 0$. Then $Q(f)(x) \neq 0$ on an open neighborhood U of x_0. Necessarily, U contained in $s(S)$. Since G is topologically principal and S does not meet $G^{(0)}$, the induced homeomorphism $\alpha_S : s(S) \to r(S)$ is not the identity map on U.

Therefore, there exists $x_1 \in U$ such that $x_2 = \alpha_S(x_1) \neq x_1$. We choose $h \in C_c(G^{(0)})$ such that $h(x_1) = 1$ and $h(x_2) = 0$. We have $hf = f(h \circ \alpha_S)$. Therefore,

$$\begin{aligned} Q(f)(x_1) &= h(x_1)Q(f)(x_1) = Q(hf)(x_1) \\ &= Q(f(h \circ \alpha_S))(x_1) = Q(f)(x_1)h(x_2) = 0. \end{aligned}$$

This is a contradiction. Therefore $Q(f) = 0$. Next, let us consider an arbitrary $f \in C_c(G, \Sigma)$ with compact support K in G. We use the fact that $G^{(0)}$ is both open and closed in G. The compact set $K \setminus G^{(0)}$ can be covered by finitely many open bisections $S_1, \ldots, S_n$ of G. Replacing if necessary S_i by $S_i \setminus G^{(0)}$, we may assume that $S_i \cap G^{(0)} = \emptyset$. We set $S_0 = G^{(0)}$. We introduce a partition of unity $(h_0, h_1, \ldots, h_n)$ subordinate to the open cover $(S_0, S_1, \ldots, S_n)$ of K: for all $i = 0, \ldots, n$, $h_i : G \to [0, 1]$ is continuous, it has a compact support contained in S_i and $\sum_{i=0}^n h_i(\gamma) = 1$ for all $\gamma \in K$. We define $f_i \in C_c(G, \Sigma)$ by $f_i(\sigma) = h_i(\dot\sigma)f(\sigma)$. Then, we have $f = \sum_{i=0}^n f_i$, $f_0 = P(f)$ and f_i has its support contained in S_i for all i. Since $f_0 \in C_0(G^{(0)})$, $Q(f_0) = f_0$. On the other hand, according to the above, $Q(f_i) = 0$ for $i = 1, \ldots, n$. Therefore, $Q(f) = f_0 = P(f)$. □

The C*-module $C_0(G^{(0)}, H)$ over $C_0(G^{(0)})$ introduced earlier to define the representation π and the reduced norm on $C_c(G, \Sigma)$ is the completion of A with respect to the B-valued inner product $P(a^*a')$; the representation π is left multiplication.

The conditional expectation P will be used to recover the elements of A as sections of the line bundle L:

Lemma 4.4. *Let (G, Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $P : A = C^*_{red}(G, \Sigma) \to B = C_0(G^{(0)})$ be the restriction map. Then we have the following formula: for all $\sigma \in \Sigma$, for all $n \in A$ such that supp$'(n)$ is a bisection containing $\dot\sigma$ and all $a \in A$:*

$$P(n^*a)(s(\sigma)) = \overline{n(\sigma)}a(\sigma).$$

Proof. This results from the definitions. □

The last property of the subalgebra $B = C_0(G^{(0)}))$ of $(A = C^*_{red}(G, \Sigma)$ which interests us is that it is regular. This requires the notion of normalizer as introduced by A. Kumjian.

Definition 4.5. [25, 1.1] Let B be a sub C*-algebra of a C*-algebra A.

(i) Its *normalizer* is the set

$$N(B) = \{n \in A : nBn^* \subset B \quad \text{and} \quad n^*Bn \subset B\}.$$

(ii) One says that B is *regular* if its normalizer $N(B)$ generates A as a C*-algebra.

Before studying the normalizer of $C_0(G^{(0)}))$ in $C^*_{red}(G,\Sigma)$, let us give some consequences of this definition. We first observe that $B \subset N(B)$ and $N(B)$ is closed under multiplication and involution. It is also a closed subset of A. We shall always assume that B contains an approximate unit of A. This condition is automatically satisfied when B is maximal abelian and A has a unit but this is not so in general (see [49]). We then have the following obvious fact.

Lemma 4.6. *Assume that B be is a sub C^*-algebra of a C^*-algebra A containing an approximate unit of A. Let $n \in N(B)$. Then $nn^*, n^*n \in B$.*

Assume also that B is abelian. Let $X = \hat{B}$ so that $B = C_0(X)$. For $n \in N(B)$, define $dom(n) = \{x \in X : n^*n(x) > 0\}$ and $ran(n) = \{x \in X : nn^*(x) > 0\}$. These are open subsets of X.

Proposition 4.7. [25, 1.6] *Given $n \in N(B)$, there exists a unique homeomorphism $\alpha_n : dom(n) \to ran(n)$ such that, for all $b \in B$ and all $x \in dom(n)$,*

$$n^*bn(x) = b(\alpha_n(x))n^*n(x).$$

Proof. See [25]. The proof uses the polar decomposition $n = u|n|$ of n in the envelopping von Neumann algebra A^{**}. The partial isomorphism of B: $b \mapsto u^*bu$ implemented by the partial isometry u gives the desired homeomorphism α_n. □

Proposition 4.8. *Let (G,Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $A = C^*_{red}(G,\Sigma)$ and $B = C_0(G^{(0)})$ be as above. Then*

(i) *If the open support $S = supp'(a)$ of $a \in A$ is a bisection of G, then a belongs to $N(B)$ and $\alpha_a = \alpha_S$;*

(ii) *If G is topologically principal, the converse is true. Namely the normalizer $N(B)$ consists exactly of the elements of A whose open support is a bisection.*

Proof. Suppose that $S = supp'(a)$ is a bisection. Then, for $b \in B$,

$$a^*ba(\sigma) = \int \overline{a(\tau\sigma^{-1})}b \circ r(\tau)a(\tau)d\lambda_{s(\sigma)}(\dot{\tau}).$$

The integrand is zero unless $\dot\tau \in S$ and $\dot{\overline{\tau\sigma^{-1}}} \in S$, which implies that $\dot\sigma$ is a unit. Therefore $supp'(a^*ba) \subset G^{(0)}$ and $a^*ba \in B$. Similarly, $aba^* \in B$. Moreover, if $\dot\sigma = x$ is a unit, we must have $\dot\tau = Sx$ and therefore

$$a^*ba(x) = a^*a(x)b \circ r(Sx) = a^*a(x)b \circ \alpha_S(x).$$

This shows that $\alpha_a = \alpha_S$.

Conversely, let us assume that a belongs to $N(B)$. Let $S = supp'(a)$. Let us fix $x \in dom(a)$. The equality

$$b(\alpha_a(x)) = \int \frac{|a(\tau)|^2}{a^*a(x)} b \circ r(\tau) d\lambda_x(\dot\tau)$$

holds for all $b \in B$. In other words, the pure state $\delta_{\alpha_a(x)}$ is expressed as a (possibly infinite) convex combination of pure states. This implies that $a(\tau) = 0$ if $r(\tau) \neq \alpha_a(x)$. Let

$$T = \{\gamma \in G : s(\gamma) \in dom(a), \text{and } r(\gamma) = \alpha_a \circ s(\gamma)\}.$$

We have established the containment $S \subset T$. This implies $SS^{-1} \subset TT^{-1} \subset G'$. If G is topologically principal, SS^{-1} which is open must be contained in $G^{(0)}$. Similarly, $S^{-1}S$ must be contained in $G^{(0)}$. This shows that S is a bisection. □

Corollary 4.9. *Let (G, Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $A = C^*_{red}(G, \Sigma)$. Then $B = C_0(G^{(0)})$ is a regular sub-C*-algebra of A.*

Proof. Since G is étale, the open bisections of G form a basis of open sets for G. Every element $f \in C_c(G, \Sigma)$ can be written as a finite sum of sections supported by open bisections. Thus the linear span of $N(B)$ contains $C_c(G, \Sigma)$. Therefore, $N(B)$ generates A as a C*-algebra. □

We continue to investigate the properties of the normalizer $N(B)$.

Lemma 4.10. [25, 1.7] *Let B be a sub-C*-algebra of a C*-algebra A. Assume that B is abelian and contains an approximate unit of A. Then*

(i) *If $b \in B$, $\alpha_b = id_{dom(b)}$.*
(ii) *If $m, n \in N(B)$, $\alpha_{mn} = \alpha_m \circ \alpha_n$ and $\alpha_{n^*} = \alpha_n^{-1}$.*

This shows that $\mathcal{G}(B) = \{\alpha_a, a \in N(B)\}$ is a pseudogroup on X. By analogy with the canonical action of the inverse semigroup of open bisections of an étale groupoid, we shall call the map $\underline{\alpha}$:

$N(B) \to \mathcal{G}(B)$ such that $\underline{\alpha}(n) = \alpha_n$ the canonical action of the normalizer.

Definition 4.11. We shall say that $\mathcal{G}(B)$ is the *Weyl pseudogroup* of (A, B). We define the *Weyl groupoid* of (A, B) as the groupoid of germs of $\mathcal{G}(B)$.

Proposition 4.12. *Let B be a sub-C^*-algebra of a C^*-algebra A. Assume that B is abelian and contains an approximate unit of A. Then:*

(i) *The kernel of the canonical action $\underline{\alpha} : N(B) \to \mathcal{G}(B)$ is the commutant $N(B) \cap B'$ of B in $N(B)$.*
(ii) *If B is maximal abelian, then $ker\underline{\alpha} = B$.*

Proof. If $n \in N(B) \cap B'$, then for all $b \in B$, $n^*bn = bn^*n$. By comparing with the definition of α_n, we see that $\alpha_n(x) = x$ for all $x \in dom(n)$. Conversely, suppose that $n \in N(B)$ satisfies $n^*bn(x) = b(x)n^*n(x)$ for all $b \in B$ and all $x \in dom(n)$. We also have $n^*bn(x) = b(x)n^*n(x) = 0$ when $x \notin dom(n)$ because of the inequality $0 \leq n^*bn \leq \|b\|n^*n$ for $b \in B_+$. Therefore $n^*bn = bn^*n$ for all $b \in B$. As observed in [25, 1.9], this implies that $(nb - bn)^*(nb - bn) = 0$ for all $b \in B$ and $nb = bn$ for all $b \in B$. The assertion (ii) is an immediate consequence of (i). □

Let us study the normalizer $N(B)$ in our particular situation, where $A = C^*_{red}(G, \Sigma)$ and $B = C_0(G^{(0)})$.

Proposition 4.13. *Let (G, Σ) be a twisted étale Hausdorff locally compact second countable groupoid. Let $A = C^*_{red}(G, \Sigma)$ and $B = C_0(G^{(0)})$ be as above. Assume that G is topologically principal. Then,*

(i) *the Weyl pseudogroup $\mathcal{G}(B)$ of (A, B) consists of the partial homeomorphisms α_S where S is an open bisection of G such that the restriction of the associated line bundle L to S is trivializable;*
(ii) *the Weyl groupoid $G(B)$ of (A, B) is canonically isomorphic to G.*

Proof. Recall that $\mathcal{S}$ denotes the inverse semigroup of open bisections of G and $\mathcal{G}$ denotes the pseudogroup defined by $\mathcal{S}$. We have defined the canonical action $\alpha : \mathcal{S} \to \mathcal{G}$ and the canonical action $\underline{\alpha} : N(B) \to \mathcal{G}(B)$. We have seen that α and $\underline{\alpha}$ are related by $\underline{\alpha} = \alpha \circ supp'$, where $supp'(n)$ denotes the open support of $n \in N(B)$, Moreover, the restriction of the line bundle to $S = supp'(n)$ is trivializable, since it possesses a non-vanishing section. Conversely, let S be an open bisection such that the restriction $L_{|S}$ is trivializable. Let us choose a non-vanishing continuous section $u : S \to L$. Replacing $u(\gamma)$ by $u(\gamma)/\|u(\gamma)\|$, we may assume that $\|u(\gamma)\| = 1$ for all $\gamma \in S$. Then, we choose $h \in C_0(G^{(0)})$ such that $supp'(h) = s(S)$ and define the section $n : G \to L$ by $n(\gamma) = u(\gamma)h \circ s(\gamma)$ if $\gamma \in S$ and $n(\gamma) = 0$ otherwise. Let (h_i) be a sequence in $C_c(G^{(0)})$, with $supp(h_i) \subset s(S)$, converging uniformly to h. Then $uh_i \in C_c(G, \Sigma)$ and the sequence (uh_i) converges to n in the norm $\|.\|_I$ introduced earlier. This implies that n belongs to A. We have $S = supp'(n)$ as desired. This shows that $\mathcal{G}(B)$ is exactly the pseudogroup consisting of the partial homeomorphisms α_S such that S is an open bisection of G on which L is trivializable. According to a theorem of Douady and Soglio-Hérault (see Appendix of [16]), for all open bisection S and all $\gamma \in S$, there exists an open neighborhood T of γ contained in S on which L is trivializable. Therefore $\mathcal{G}(B)$ and the pseudogroup $\mathcal{G}$ defined by all open bisections have the same groupoid of germs, which is isomorphic to G by Corollary 3.6. $\square$

Let us see next how the twist Σ over G can be recovered from the pair (A, B). This is done exactly as in Section 3 of [25]. Given an abstract pair (A, B), we set $X = \hat{B}$ and introduce

$$D = \{(x, n, y) \in X \times N(B) \times X : n^*n(y) > 0 \text{ and } x = \alpha_n(y)\}$$

and its quotient $\Sigma(B) = D/\sim$ by the equivalence relation: $(x, n, y) \sim (x', n', y')$ if and only if $y = y'$ and there exist $b, b' \in B$ with $b(y), b'(y) > 0$ such that $nb = n'b'$. The class of (x, n, y) is denoted by $[x, n, y]$. Now $\Sigma(B)$ has a natural structure of groupoid over X, defined exactly in the same fashion as a groupoid of germs: the range and source maps are defined by $r[x, n, y] = x, s[x, n, y] = y$, the product by $[x, n, y][y, n', z] = [x, nn', z]$ and the inverse by $[x, n, y]^{-1} = [y, n^*, x]$.

The map $(x, n, y) \to [x, \alpha_n, y]$ from D to $G(B)$ factors through the quotient and defines a groupoid homomorphism from $\Sigma(B)$ onto $G(B)$. Moreover the subset $\mathcal{B} = \{[x, b, x] : b \in B, b(x) \neq 0\} \subset \Sigma(B)$ can be identified with the trivial group bundle $\mathbf{T} \times X$ via the map $[x, b, x] \mapsto (b(x)/|b(x)|, x)$. In general, $\mathcal{B} \to \Sigma(B) \to G(B)$ is not an extension, but this is the case when B is maximal abelian.

Proposition 4.14. *Assume that B is a masa in A containing an approximate unit of A. Then*

$$\mathcal{B} \to \Sigma(B) \to G(B)$$

is (algebraically) an extension.

Proof. We have to check that an element $[x, n, y]$ of $\Sigma(B)$ which has a trivial image in $G(B)$ belongs to $\mathcal{B}$. If the germ of α_n at y is the identity, then $x = y$ and we have a neighborhood U of y contained in $dom(n)$ such that $\alpha_n(z) = \alpha_{n^*}(z) = z$ for all $z \in U$. We choose $b \in B$ with compact support contained in U and such that $b(x) > 0$ and we define $n' = nb$. Then $\alpha_{n'}$ is trivial. According to Proposition 4.12, n' belongs to B and $[x, n, x] = [x, n', x]$ belongs to $\mathcal{B}$. □

We shall refer to $\Sigma(B)$ as the *Weyl twist* of the pair (A, B).

Proposition 4.15. *Let (G, Σ) be a twisted étale Hausdorff locally compact second countable topologically principal groupoid. Let $A = C^*_{red}(G, \Sigma)$ and $B = C_0(G^{(0)})$ be as above. Then we have a canonical isomorphism of extensions:*

$$\begin{array}{ccccc}
\mathcal{B} & \longrightarrow & \Sigma(B) & \longrightarrow & G(B) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{T} \times G^{(0)} & \longrightarrow & \Sigma & \longrightarrow & G
\end{array}$$

Proof. The left and right vertical arrows have been already defined and shown to be isomorphisms. It suffices to define the middle vertical arrow and show that it is a groupoid homomorphism which makes the diagram commutative. Let $(x, n, y) \in D$. Since n belongs to $N(B)$ and G is topologically principal, $S = supp'(n)$ is an open bisection of G. The element $n(Sy)/\sqrt{n^*n(y)}$ is a unitary element of L because $n^*n(y) = \|n(Sy)\|^2$ and can therefore be viewed as an element of Σ. Let $(x, n', y) \sim (x, n, y)$. There exist $b, b' \in B$ with $b(y), b'(y) > 0$ such that $nb = n'b'$. This implies that the open supports $S = supp'(n)$ and $S = supp'(n)$ agree on some neighborhood of Sy. In particular, $Sy = S'y$. Moreover, the equality $n(Sy)b(y) = n'(Sy)b'(y)$ implies that $n(Sy)/\sqrt{n^*n(y)} = n'(Sy)/\sqrt{n'^*n'(y)}$. Thus we have a well-defined map $\Phi : [x, n, y] \mapsto (n(Sy)/\sqrt{n^*n(y)}, Sy)$ from $\Sigma(B)$ to Σ. Let us check that it is a groupoid homomorphism.

Suppose that we are given $(x,m,y),(y,n,z)\in D$. Let $S=supp'(m)$, $T=supp'(n)$. Then $supp'(mn)=ST$. We have to check the equality

$$\frac{mn(STz)}{\sqrt{(mn)^*mn(z)}}=\frac{m(Sy)}{\sqrt{m^*m(y)}}\frac{n(Tz)}{\sqrt{n^*n(z)}}.$$

It is satisfied because $mn(STz)=m(Sy)n(Tz)$ and

$$\begin{aligned}(mn)^*(mn)(z)&=(n^*(m^*m)n)(z)\\&=(m^*m)(\alpha_n(z))n^*n(z)=m^*m(y)n^*n(z).\end{aligned}$$

The image of $[x,n,y]^{-1}=[y,n^*,x]$ is

$$n^*(S^{-1}x)/\sqrt{nn^*(x)}=(n(xS))^*/\sqrt{nn^*(x)}.$$

It is the inverse of $n(Sy)/\sqrt{n^*n(y)}$ because $xS=Sy$ and $nn^*(x)=n^*n(y)$ and the involution agrees with the inverse on $\Sigma\subset L$. Let us check that we have a commutative diagram. The restriction of Φ to $\mathcal{B}$ sends $[x,b,x]$ to $(b(x)/|b(x)|,x)$. This is exactly the left vertical arrow. The image of $[x,n,y]$ in $G(B)$ is the germ $[x,\alpha_n,y]$. The image of $(n(Sy)/\sqrt{n^*n(y)},Sy)$ in G is Sy. The map $[x,\alpha_n,y]\mapsto Sy$ is indeed the canonical isomorphism from $G(B)$ onto G. $\square$

In the previous proposition, we have viewed $\Sigma(B)$ as an algebraic extension. It is easy to recover the topology of $\Sigma(B)$. Indeed, as we have already seen, every $n\in N(B)$ defines a trivialization of the restriction of $\Sigma(B)$ to the open bisection $S=supp'(n)$. This holds in the abstract framework. Assume that B is a masa in A. Let $n\in N(B)$. Its open support is by definition the open bisection $S\subset G(B)$ which induces the same partial homeomorphism as n. We define the bijection

$$\varphi_n:\mathbf{T}\times dom(n)\to\Sigma(B)_{|S},$$

by $\varphi_n(t,x)=[\alpha_n(x),tn,x]$.

Lemma 4.16. (*cf.* [25, Section 3]) *Assume that B is a masa in A containing an approximate unit of A. With above notation,*

(i) *Two elements $n_1,n_2\in N(B)$ which have the same open support S define compatible trivializations of $\Sigma(B)_{|S}$.*
(ii) *$\Sigma(B)$ is a locally trivial topological twist over $G(B)$.*

Proof. For (i), assume that n_1 and n_2 have the same open support S. Then, according to Proposition 4.12, there exist $b_1,b_2\in B$, non vanishing on $s(S)$ and such that $n_1b_1=n_2b_2$. A simple computation from the relation $\varphi_{n_1}(t_1, x) = \varphi_{n_2}(t_2, x)$ and the fact that for $n \in N(B)$ and $b \in B$, the equality $nb = 0$ implies $b(x) = 0$ whenever $n^*n(x) > 0$ gives $t_2 = t_1 u(x)$ where $u(x) = \frac{b_2(x)|b_1(x)|}{|b_2(x)|b_1(x)}$. Therefore, the transition function is a homeomorphism. We deduce (ii). Indeed, we have given a topology to $\Sigma(B)_{|S}$ whenever S is a bisection arising from the Weyl pseudogroup $\mathcal{G}(B)$. This family, which is stable under finite intersection and which covers $\Sigma(B)$, is a base of open sets for the desired topology. □

5. Cartan Subalgebras in C*-Algebras

Motivated by the properties of the pair $\big(A = C^*_{red}(G, \Sigma),\ B = C_0(G^{(0)})\big)$ arising from a twisted étale locally compact second countable Hausdorff topologically principal groupoid, we make the following definition, analogous to [18, Definition 3.1] of a Cartan subalgebra in a von Neumann algebra. We shall always assume that the ambient C*-algebra A is separable.

Definition 5.1. We shall say that an abelian sub-C*-algebra B of a C*-algebra A is a *Cartan subalgebra* if

(i) B contains an approximate unit of A;
(ii) B is maximal abelian;
(iii) B is regular;
(iv) there exists a faithful conditional expectation P of A onto B.

We shall say that (A, B) is a *Cartan pair* when B is a Cartan subalgebra.

Let us give some comments about the definition. First, when A has a unit, a maximal abelian sub-C*-algebra necessarily contains the unit; however, as said earlier, there exist maximal abelian sub-C*-algebras which do not contain an approximate unit for the ambient C*-algebra. Since in our models, namely étale groupoid C*-algebras, the subalgebra corresponding to the unit space always contains an approximate unit of A, we have to make this assumption. Second this definition of a Cartan subalgebra should be compared to the Definition 1.3 of a C*-diagonal given by A. Kumjian in [25] (see also [41]): there it is assumed that B has the unique extension property, a property introduced by J. Anderson and studied by R. Archbold et al. If B has the unique extension property (and under the assumption that it contains an approximate unit of A), it is maximal abelian and there exists one and only one conditional expectation onto B. We shall say more about the unique extension property when we compare Theorem 5.9 and Kumjian's theorem. The analysis of the previous section can be summarized by the following result.

Theorem 5.2. *Let (G,Σ) be a twisted étale Hausdorff locally compact second countable topologically principal groupoid. Then $C_0(G^{(0)})$ is a Cartan subalgebra of $C^*_r(G,\Sigma)$.*

Given a Cartan pair (A,B), we construct the normalizer $N(B)$, the Weyl groupoid $G(B)$ on $X=\hat{B}$, the Weyl twist $\Sigma(B)$ and the associated line bundle $L(B)$. In fact, these constructions can be made under the sole assumption that B is a masa. Let us see how the elements of A define sections of the line bundle $L(B)$ or equivalently, functions $f:\Sigma\to\mathbf{C}$ satisfying $f(t\sigma)=\bar{t}f(\sigma)$ for all $t\in\mathbf{T}$ and $\sigma\in\Sigma(B)$. The answer is given by Lemma 4.4 (this formula also appears in [25]). Recall that $\Sigma(B)$ is defined as a quotient of

$$D=\{(x,n,y)\in X\times N(B)\times X : n^*n(y)>0 \text{ and } x=\alpha_n(y)\}.$$

Lemma 5.3. *Given $a\in A$ and $(x,n,y)\in D$, we define*

$$\hat{a}(x,n,y)=\frac{P(n^*a)(y)}{\sqrt{n^*n(y)}}.$$

Then

(i) *$\hat{a}(x,n,y)$ depends only on its class in $\Sigma(B)$;*
(ii) *$\hat{a}$ defines a continuous section of the line bundle $L(B)$;*
(iii) *the map $a\mapsto\hat{a}$ is linear and injective.*

Proof. Assertion (i) is clear, since $\hat{a}(x,nb,y)=\hat{a}(x,n,y)$ for all $b\in B$ such that $b(y)>0$. For (ii), the equality $\hat{a}(x,tn,y)=\bar{t}\hat{a}(x,n,y)$ for all $t\in\mathbf{T}$ shows that $\hat{a}$ defines a section of $L(B)$. To get the continuity, it suffices to check the continuity of $\hat{a}$ on the open subsets $\Sigma(B)_{|S}$, where S is the open support of $n\in N(B)$. But this is exactly the continuity of the function $y\mapsto P(n^*a)(y)/\sqrt{n^*n(y)}$ on $dom(n)$. The linearity in (iii) is clear. Let us assume that $\hat{a}=0$. Let $n\in N(B)$. Then $P(n^*a)(y)=0$ for all $y\in dom(n)$, hence also in its closure. If y does not belong to the closure of $dom(n)$, we can find $b\in B$ such that $b(y)=1$ and $nb=0$. Then $P(n^*a)(y)=P(b^*n^*a)(y)=0$. Therefore $P(n^*a)=0$ for all $n\in N(B)$. By regularity of B, this implies $P(a^*a)=0$. By faithfulness of P, this implies that $a=0$. □

Definition 5.4. The map $\Psi : a \mapsto \hat{a}$ from A to the space of continuous sections of $L(B)$ will be called the *evaluation map* of the Cartan pair (A, B).

Lemma 5.5. *Let (A, B) be a Cartan pair. For $n \in N(B)$ and $x \in dom(n)$ such that the germ of α_n at x is not trivial, we have $P(n)(x) = 0$.*

Proof. Since the germ of α_n at x is not trivial, there exists a sequence (x_i) in $dom(n)$ which converges to x and such that $\alpha_n(x_i) \neq x_i$. We fix i. There exist $b', b'' \in B$ such that $b'(x_i) = 1$, $b''(x_i) = 0$ and $b''n = nb'$. Indeed, there exists $b \in B$ with compact support contained in $ran(n)$ such that $b(\alpha_n(x_i))(n^*n)(x) = 1$ and $b(x_i) = 0$. Then $b' = (b \circ \alpha_n)(n^*n)$ and $b'' = (nn^*)b$ satisfy the conditions. We have

$$\begin{aligned} P(n)(x_i) &= P(n)(x_i)b'(x_i) = P(nb')(x_i) \\ &= P(b''n)(x_i) = b''(x_i)P(n)(x_i) = 0. \end{aligned}$$

By continuity of $P(n)$, $P(n)(x) = 0$. □

Corollary 5.6. *Let $a \mapsto \hat{a}$ be the evaluation map of the Cartan pair (A, B).*

(i) *Suppose that b belongs to B; then $\hat{b}$ vanishes off X and its restriction to X is its Gelfand transform.*
(ii) *Suppose that n belongs to $N(B)$; then the open support of $\hat{n}$ is the open bisection of $G(B)$ defined by the partial homeomorphism α_n.*

Proof. Let us show (i). If $\gamma = [\alpha_n(x), \alpha_n, x] \in G(B)$ is not a unit, the germ of α_n at x is not trivial. According to the lemma, for all $b \in B$, $P(n^*b)(x) = P(n)(x)b(x) = 0$. Therefore, $\hat{b}(\gamma) = 0$. On the other hand, if $\gamma = x$ is a unit, $\hat{b}(x) = P({b_1}^*b)(x) = {b_1}^*(x)b(x) = b(x)$ for $b_1 \in B$ such that $b_(x) = 1$. Let us show (ii). If $n \in N(B)$, the lemma shows that $\hat{n}[x, m, y] = 0$ unless $y \in dom(n)$ and α_m has the same germ as α_n at y. Then $[x, \alpha_m, y] = [x, \alpha_n, y]$ belongs to the open bisection S_n of $G(B)$ defined by the partial homeomorphism α_n. On the other hand, $\hat{n}(x, n, y) = n^*n(y)/\sqrt{n^*n(y)}$ is non zero for $y \in dom(n)$. □

Proposition 5.7. *The Weyl groupoid $G(B)$ of a Cartan pair (A, B) is a Hausdorff étale groupoid .*

Proof. By construction as a groupoid of germs, $G(B)$ is étale. Let us show that the continuous functions $\hat{a}$, where $a \in A$ separate the points of $G(B)$ in the sense that for all $\sigma, \sigma' \in \Sigma$ such that $\dot{\sigma} \neq \dot{\sigma}'$, there exists $a \in A$ such that $\hat{a}(\sigma) \neq 0$ and $\hat{a}(\sigma') = 0$. By construction, $\sigma = [x, n, y], \sigma' = [x', n', y']$ where $n, n' \in N(B)$ and $y \in dom(n), y' \in dom(n')$. If $y \neq y'$, we can take a of the form nb where $b(y) \neq 0$ and $b(y') = 0$. If $y = y'$, since α_n and $\alpha_{n'}$ do not have the same germ at y, we have by Lemma 5.5 that $P(n'^* n)(y) = 0$, which implies $\hat{n}(\sigma') = 0$. On the other hand, $\hat{n}(\sigma) = \sqrt{n^* n(y)}$ is non-zero. We can furthermore assume that $\hat{a}(\sigma) = 1$. Let $U = \{\tau : |\hat{a}(\tau) - 1| < 1/2\}$ and $V = \{\tau : |\hat{a}(\tau)| < 1/2\}$. Their images $\dot{U}, \dot{V}$ in $G(B)$ are open, disjoint and $\dot{\sigma} \in \dot{U}, \dot{\sigma}' \in \dot{V}$. □

Lemma 5.8. *Let (A, B) be a Cartan pair. Let $N_c(B)$ be the set of elements n in $N(B)$ such $\hat{n}$ has compact support and let A_c be its linear span. Then*

(i) *$N_c(B)$ is dense in $N(B)$ and A_c is dense in A;*
(ii) *the evaluation map $\Psi : a \mapsto \hat{a}$ defined above sends bijectively A_c onto $C_c(G(B), \Sigma(B))$ and $B_c = B \cap A_c$ onto $C_c(G^{(0)})$;*
(iii) *the evaluation map $\Psi : A_c \to C_c(G(B), \Sigma(B))$ is a $*$-algebra isomorphism.*

Proof. For (i), given $n \in N(B)$, there exists $b \in B$ such that $nb = n$. There exists a sequence (b_i) in B such that $\hat{b}_i \in C_c(G^{(0)})$ and (b_i) converges to b. Then nb_i belongs to $N_c(B)$ and the sequence (nb_i) converges to n. Note that $N_c(B)$ is closed under product and involution and that A_c is a dense sub-$*$-algebra of A. Let us prove (ii). By construction, $\Phi(A_c)$ is contained in $C_c(G, \Sigma)$. The injectivity of Φ has been established in Lemma 5.4. Let us show that $\Phi(A_c) = C_c(G, \Sigma)$. The family of open bisections $S_n = \{[\alpha_n(x), \alpha_n, x], x \in dom(n)\}$, where n runs over $N(B)$, forms an open cover of $G(B)$. If $f \in C_c(G, \Sigma)$ has its support contained in S_n, then $\hat{n}$ is a non-vanishing continuous section over S_n and there exists $h \in C_c(G^{(0)})$ such that $f = \hat{n}h$. Since $h = \hat{b}$ with $b \in B_c$, $f = \hat{a}$, where $a = nb$ belongs to $N_c(B)$. For a general $f \in C_c(G, \Sigma)$, we use a partition of unity subordinate to a finite open cover $S_{n_1}, \ldots, S_{n_l}$ of the support of f. Let us prove (iii). By linearity of Ψ, it suffices to check the relations $\Psi(mn) = \Psi(m) * \Psi(n)$ and $\Psi(n^*) = \Psi(n)^*$ for $m, n \in N(B)$. According to Corollary 5.6, $\Psi(mn)(\sigma) = 0$ unless

$\sigma = t[x, mn, z]$ with $z \in dom(mn)$ and $t \in \mathbf{T}$; then we have

$$\Psi(mn)(t[x, mn, z]) = \bar{t}\sqrt{((mn)^*mn)(z)}.$$

On the other hand, $\Psi(m)\Psi(n)(\sigma) = 0$ unless σ is of the form

$$\sigma = t[x, m, y][y, n, z] = t[x, mn, z]$$

and then

$$\begin{aligned}\Psi(m)\Psi(n)(t[x, mn, z]) &= \Psi(m)(t[x, m, y])\Psi(n)([y, n, z] \\ &= \bar{t}\sqrt{(m^*m)(y)(n^*n)(z)}.\end{aligned}$$

The equality results from

$$\begin{aligned}(mn)^*(mn)(z) &= (n^*(m^*m)n)(z) \\ &= (m^*m)(\alpha_n(z))n^*n(z) = m^*m(y)n^*n(z).\end{aligned}$$

Similarly, $\Psi(n^*)(\sigma) = 0$ unless $\sigma = \bar{t}[y, n^*, x]$ with $x \in dom(n^*)$ and $t \in \mathbf{T}$ and then we have

$$\Psi(n^*)(\bar{t}[y, n^*, x]) = t\sqrt{(nn^*)(x)}.$$

On the other hand, $\Psi(n)^*(\sigma) = \overline{\Psi(n)(\sigma^{-1})} = 0$ unless $\sigma^{-1} = t[x, n, y]$ with $y \in dom(n)$ and $t \in \mathbf{T}$ and then we have

$$\Psi(n)^*(\bar{t}[y, n^*, x]) = t\sqrt{(n^*n)(y)}.$$

These numbers are equal because $nn^*(x) = n^*n(y)$. □

Theorem 5.9. *Let B be a Cartan sub-algebra of a separable C^*-algebra A. Then*

(i) *there exists a twist (G, Σ) where G is a second countable locally compact Hausdorff, topologically principal étale groupoid and an isomorphism of $C^*_r(G, \Sigma)$ onto A carrying $C_0(G^{(0)})$ onto B;*

(ii) *the above twist is unique up to isomorphism; it is isomorphic to the Weyl twist $(G(B), \Sigma(B))$.*

Proof. Let $(G, \Sigma) = (G(B), \Sigma(B))$. Let us show that the evaluation map $\Psi : A_c \to C_c(G, \Sigma)$ is an isometry with respect to the norms of A and $C^*_r(G, \Sigma)$. Since P is faithful, we have for any $a \in A$ the equality

$$\|a\| = \sup\{\|P(c^*a^*ac)\|^{1/2} : c \in A_c, P(c^*c) \le 1\}.$$

If we assume that a belongs to A_c, then $\hat{a}$ belongs to $C_r^*(G,\Sigma)$ and satisfies a similar formula:

$$\begin{aligned}\|\hat{a}\| &= \sup\{\|\hat{P}(f^*(\hat{a})^*\hat{a}f)\|^{1/2} : f \in C_c(G,\Sigma), \hat{P}(f^*f) \le 1\}\\ &= \sup\{\|\hat{P}((\hat{c})^*(\hat{a})^*\hat{a}\hat{c})\|^{1/2} : c \in A_c, \hat{P}((\hat{c})^*\hat{c}) \le 1\}.\end{aligned}$$

Since $\Psi : A_c \to C_c(G,\Sigma)$ satisfies the relation $\hat{P} \circ \Psi = \Psi \circ P$, where $\hat{P}$ is the restriction map to $\hat{B}$, we have the equality of the norms: $\|\hat{c}\| = \|c\|$. Hence Ψ extends to a C*-algebra isomorphism $\tilde{\Psi} : A \to C_r^*(G,\Sigma)$. By continuity of point evaluation, $\tilde{\Psi}(a) = \Psi(a)$ as defined initially. Therefore, the evaluation map is a C*-algebra isomorphism of of $C_r^*(G,\Sigma)$ onto A carrying $C_0(G^{(0)})$ onto B. The separability of $C_r^*(G,\Sigma)$ implies that of $C_0(G,E)$. One deduces that G is second countable. Since $G = G(B)$ is a groupoid of germs, it results from Proposition 3.6 that G is topologically principal. The uniqueness of the twist up to isomorphism has been established in Proposition 4.15. □

We have mentioned earlier that the unique extension property of B implies the uniqueness of the conditional expectation onto B. The uniqueness still holds for Cartan subalgebras.

Corollary 5.10. *Let B be a Cartan subalgebra of a C*-algebra A. Then, there exists a unique expectation onto B.*

Proof. This results from the above theorem and Proposition 4.3. □

The following proposition is essentially a reformulation of Kumjian's theorem (see [25] and [41]). For the sake of completeness, we recall his proof. One says that the subalgebra B has the unique extension property if every pure state of B extends uniquely to a (pure) state of A. A C*-diagonal is a Cartan subalgebra which has the unique extension property.

Proposition 5.11. (*cf.* [25], [41]) *Let (A,B) be a Cartan pair. Then B has the unique extension property if and only if the Weyl groupoid $G(B)$ is principal.*

Proof. We may assume that $(A,B) = (C_r^*(G,\Sigma), C_0(G^{(0)}))$, where G is an étale topologically principal Hausdorff groupoid and Σ is a twist over G. Suppose that G is principal. A. Kumjian shows that this implies that the linear span of the set $N_f(B)$ of free normalizers is dense in the kernel of the conditional expectation P, where a normalizer $n \in N(B)$ is said to be free if $n^2 = 0$. Indeed, since an arbitrary element of the kernel can be approximated by elements in $C_c(G,\Sigma)\cap Ker(P)$, it suffices to consider a continuous section f with compact support which vanishes on $G^{(0)}$. Since the compact support of f does not meet the diagonal $G^{(0)}$, which is both open and closed, it admits a finite cover by open bisections U_i such that $r(U_i)\cap s(U_i)=\emptyset$. Let (h_i) be a partition of unity subordinate to the open cover (U_i). Then, $f=\sum g_i$, where $g_i(\sigma)=f(\sigma)h_i(\dot\sigma)$ is a free normalizer. Then, he observes that free normalizers are limits of commutators $ab-ba$, with $a\in A$ and $b\in B$. This show that $A=B+\overline{span}[A,B]$, which is one of the characterizations of the extension property given in Corollary 2.7 of [2]. We suppose now that B has the unique extension property and we show that the isotropy of G is reduced to $G^{(0)}$. It suffices to show that for $n\in N(B)$ and $x\in dom(n)$, the equality $\alpha_n(x)=x$ implies that the germ of α_n at x is trivial. According to Lemma 5.5, it suffices to show that $P(n)(x)\neq 0$. Given $n\in N(B)$ and $x\in dom(n)$, the states $x\circ P$ and $\alpha_n(x)\circ P$ are unitarily equivalent and their transition probability ([43]) is $\frac{|P(n)(x)|^2}{n^*n(x)}$. Indeed, let (H,ξ,π) be the GNS triple constructed from the state $x\circ P$. By construction, $x\circ P$ is the state defined by the representation π and the vector ξ. On the other hand, $\alpha_n(x)\circ P$ is the state of A defined by π and the vector $\eta=\pi(u)\xi$, where u is the partial isometry of the polar decomposition $n=u|n|$ of n in A^{**}. To show that, one checks the straightforward relation $b(\alpha_n(x))=(\eta,\pi(b)\eta)$ for $b\in B$ and one uses the unique extension property. The transition probability can be computed by the formula $|(\xi,\eta)|^2=\frac{|P(n)(x)|^2}{n^*n(x)}$. If $\alpha_n(x)=x$, the transition probability is 1. In particular, $P(n)(x)\neq 0$. □

6. Examples of Cartan Subalgebras in C*-Algebras

6.1. **Crossed products by discrete groups.** In his pioneering work [50] on crossed product C* and W*-algebras by discrete groups, G. Zeller-Meier gives the following necessary and sufficient condition (Proposition 4.14) for B to be maximal abelian in the reduced crossed product $C^*_r(\Gamma;B;\sigma)$, where Γ is a discrete group acting by automorphisms on a commutative C*-algebra B and σ is a 2-cocycle: the action of Γ on $X=\hat B$ must be topologically free, meaning that for all $s\in\Gamma\setminus\{e\}$, the set $X_s=\{x\in X : sx=x\}$ must have an empty interior in X. This amounts to the groupoid $G=\Gamma\ltimes X$ of the action being topologically principal. Proposition 2.4.7 of [40] extends this result. Note that G is principal if and only if the action is free, in the sense that for all $s \in \Gamma \setminus \{e\}$, the set $X_s = \{x \in X : sx = x\}$ is empty. The particular case of the group $\Gamma = \mathbf{Z}$ is well studied (see for example [45]) and we consider only this case below.

Irrational rotations and minimal homeomorphisms of the Cantor space are examples of free actions. The C*-algebras of these dynamical systems are well understood and completely classified. I owe to I. Putnam the remark that the C*-algebra of a Cantor minimal system may contain uncountably many non-conjugate Cartan subalgebras (which are in fact diagonals in the sense of Kumjian). Indeed, according to [19], such a C*-algebra depends only, up to isomorphism, on the strong orbit equivalence class of the dynamical system; however, two minimal Cantor systems which are strongly orbit equivalent need not be flip conjugate (flip conjugacy amounts to groupoid isomorphism). More precisely, Boyle and Handelman show in [5] that the strong orbit equivalence class of the dyadic adding machine contains homeomorphisms of arbitrary entropy. These will give the same C*-algebra but the corresponding Cartan subalgebras will not be conjugate.

On the other hand, two-sided Bernoulli shifts are examples of topologically free actions which are not free. They provide examples of Cartan subalgebras which do not have the extension property. In [45], J. Tomiyama advocates the view that in relation with operator algebras, the notion of topologically free action, rather than that of free action, is the counterpart for topological dynamical systems of the notion of free action for measurable dynamical systems. The comparison of Theorem 5.9 and of the Feldman–Moore theorem completely supports this view.

6.2. **AF Cartan subalgebras in AF C*-algebras.** Approximately finite dimensional (AF) C*-algebras have privileged Cartan subalgebras. These are the maximal abelian subalgebras obtained by the diagonalization method of Strătilă and Voiculescu ([44]). In that case, the twist is trivial and the whole information is contained in the Weyl groupoid. The groupoids which occur in that fashion are the AF equivalence relations. These are the equivalence relations R on a totally disconnected locally compact Hausdorff space X which are the union of an increasing sequence of proper equivalence relations (R_n). The proper relations R_n are endowed with the topology of $X \times X$ and R is endowed with the inductive limit topology. As shown by Krieger in [23], AF C*-algebras and AF equivalence relations share the same complete invariant, namely the dimension group. One deduces that these privileged Cartan subalgebras, also called AF Cartan subalgebras, are conjugate by an automorphism of the ambient AF algebra. However, AF C*-algebras may contain other Cartan subalgebras. An example of a Cartan subalgebra in an AF C*-algebra without the unique extension property is given in [40, III.1.17]. A more striking example is given by B. Blackadar in [4]. He constructs a diagonal in the CAR algebra whose spectrum is not totally disconnected. More precisely, he realizes the CAR algebra as the crossed product $C(X)\rtimes\Gamma$ where $X = \mathbf{S}^1 \times$ Cantor space and Γ is a locally finite group acting freely on X. Note that the groupoid $X\rtimes\Gamma$ is also an AP equivalence relation, in the sense that it is the union of an increasing sequence of proper equivalence relations (R_n).

6.3. **Cuntz-Krieger algebras and graph algebras.** The Cuntz algebra $\mathcal{O}_d$ is the prototype of a C*-algebra which has a natural Cartan subalgebra without the unique extension property. By definition, $\mathcal{O}_d$ is the C*-algebra generated by d isometries $S_1, \dots, S_d$ such that $\sum_{i=1}^d S_iS_i^* = 1$. The Cartan subalgebra in question is the sub C*-algebra $\mathcal{D}$ generated by the range projections of the isometries $S_{i_1}\dots S_{i_n}$. It can be checked directly that $\mathcal{D}$ is a Cartan subalgebra of $\mathcal{O}_d$; however, it is easier to show first that $(\mathcal{O}_d, \mathcal{D})$ is isomorphic to $(C^*(G), C(X))$, where $X = \{1, \dots, d\}^{\mathbf{N}}$ and $G = G(X, T)$ is the groupoid associated to the one-sided shift $T : X \to X$ (see [40, 11, 42]):

$$G = \{(x, m - n, y) : x, y \in X, m, n \in \mathbf{N}, T^m x = T^n y\}.$$

This groupoid is not principal but it is topologically principal. In fact, the groupoid $G(X, T)$ associated to the local homeomorphism $T : X \to X$ is topologically principal if and only if T is topologically free, meaning that for all pairs of distinct integers (m, n), the set $X_{m,n} = \{x \in X : T^m x = T^n x\}$ must have an empty interior in X.

Condition (I) introduced by Cuntz and Krieger in their fundamental work [10] ensures that the subalgebra $\mathcal{D}_A$ is a Cartan subalgebra of $\mathcal{O}_A$. Here, A is a $d \times d$ matrix with entries in $\{0, 1\}$ and non-zero rows and columns. The associated dynamical system is the one-sided subshift of finite type (X_A, T_A); condition (I) guarantees that this system is topologically free. In subsequent generalizations, in terms of infinite matrices in [14] and in terms of graphs in [29], exit condition (L) replaces condition (I). On the topological dynamics side, it is a necessary and sufficient condition for the relevant groupoid to be topologically principal. On the C*-algebraic side, it is the condition which ensures that the natural diagonal subalgebra $\mathcal{D}$ is maximal abelian, hence a Cartan subalgebra. Moreover, it results from [29] that this subalgebra has the extension property if and only if the graph contains no loops. Condition (II) of [9] or its generalization (K) in [28] implies that each reduction of the groupoid to an invariant closed subset is topologically principal and therefore that the image of $\mathcal{D}$ in the corresponding quotient is still maximal abelian.

6.4. **Cartan subalgebras in continuous-trace C*-algebras.** Let us first observe that a Cartan subalgebra of a continuous-trace C*-algebra necessarily has the unique extension property. The proof given in [15, Théorème 3.2] for foliation C*-algebras is easily adapted.

Proposition 6.1. *Let B be a Cartan subalgebra of a continuous-trace C*-algebra A. Then B has the unique extension property.*

Proof. From the main theorem, we can assume that $(A, B) = (C_r^*(G, \Sigma), C_0(G^{(0)}))$, where G is an étale topologically principal Hausdorff groupoid and Σ is a twist over G. Since A is nuclear, we infer from [1, 6.2.14, 3.3.7] that G is topologically amenable and from [1, 5.1.1] that all its isotropy subgroups are amenable. Since A is CCR, we infer from [6, Section 5,] that $G^{(0)}/G$ injects continuously in $\hat{A}$ and that all the orbits of G are closed (the presence of a twist does not affect this result nor its proof). Since G is étale, these closed orbits are discrete. Now, each $h \in C_c(G^{(0)})$ belongs to the Pedersen ideal $K(A)$. Therefore, it defines a continuous function on $\hat{A}$ whose value at $[x] \in G^{(0)}/G$ is

$$\overline{h}[x] = \sum_{y \in [x]} h(y).$$

Suppose that $G(x)$ is not reduced to $\{x\}$. Then there exists an open neighborhood V of x such that $[x] \cap V = \{x\}$ and $[y] \cap V$ contains at least two elements for $y \neq x$. For $h \in C_c(G^{(0)})$ supported in V and equal to 1 on a neighborhood of x, we would obtain $\overline{h}[x] = 1$ and $\overline{h}[y] \geq 2$ for y close to x, which contradicts the continuity of $\overline{h}$. Hence G is principal and B has the unique extension property. $\square$

When a Cartan subalgebra B of a continuous-trace C*-algebra A exists, the cohomology class $[\Sigma(B)]$ of its twist is essentially the Dixmier–Douady invariant of A. Indeed, just as in the group case, the groupoid extension $\Sigma(B)$ defines an element of the cohomology group $H^2(G(B), \mathbf{T})$ (see [46] for a complete account of groupoid cohomology). Since $G(B)$ is equivalent to $\hat{B}/G(B) = \hat{A}$, this can be viewed as an element of $H^2(\hat{A}, \mathcal{T})$, where $\mathcal{T}$ is the sheaf of germs of $\mathbf{T}$-valued continuous functions. Its identification with the Dixmier–Douady invariant is done in in [25, 41, 39]. Moreover, a simple construction shows that every Čech cohomology class in $H^3(T, \mathbf{Z})$, where T is a locally compact Hausdorff space, can be realized as the Dixmier–Douady invariant of a continuous-trace C*-algebra of the above form $C^*(G, \Sigma)$.

However, Cartan subalgebras B of a continuous-trace C*-algebra A do not always exist. It has been observed (see [2, Remark 3.5.(iii)]) that there exist non-trivial n-homogeneous C*-algebras which do not have a masa with the unique extension property. Therefore, these C*-algebras do not have Cartan subalgebras. In [24, Appendix], T. Natsume gives an explicit example. Given a Hilbert bundle H over a compact space T, let us denote by A_H the continuous-trace C*-algebra defined by H. Let B be a Cartan subalgebra of A_H. The inclusion map gives a map $\hat{B} \to T$ which is a local homeomorphism and a surjection. If T is connected and simply connected, this is a trivial covering map and B decomposes as a direct sum of summands isomorphic to $C(T)$. Therefore H decomposes as a direct sum of line bundles. This is not always possible. For example there exists a vector bundle of rank 2 on the sphere $\mathbf{S}^4$ which cannot be decomposed into a direct sum of line bundles.

6.5. **Concluding remarks.** Just as in the von Neumann setting, the notion of Cartan subalgebra in C*-algebras provides a bridge between the theory of dynamical systems and the theory of operator algebras. Examples show the power of this notion, in particular to understand the structure of some C*-algebras, but also its limits. This notion has to be modified if one wants to include the class of the C*-algebras of non-Hausdorff topologically principal étale groupoids. In the case of continuous-trace C*-algebras, we have seen that the twist attached to a Cartan subalgebra is connected with the Dixmier–Douady invariant. It would be interesting to investigate its C*-algebraic significance in other situations.

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