The Hilbert Transform and Fine Continuity

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Abstract. It is shown that the Hilbert transform of a function having bounded variation in a finite interval $[c, d]$ has fine continuity properties at points in $[c, d]$ outside certain exceptional sets.

1. Introduction

The *Hilbert transform* of a function $f \in L(\mathbb{R})$ is defined by

$$\mathcal{H}f(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}\, dt, \quad x \in \mathbb{R}.$$

A question arises immediately concerning the existence of $\mathcal{H}f$, and we note that it was shown independently by Besicovitch and Kolmogoroff in the 1920s that $\mathcal{H}f$ is finite a.e. in $\mathbb{R}$. It is also natural to ask whether, or to what extent, the operator $f \to \mathcal{H}f$ preserves properties of f, such as continuity, and it is this question we are concerned with here. A well-known result of Privalov (see [14, p. 121]) asserts that, if $f \in \mathrm{Lip}_\alpha(c, d)$, i.e. $f(x+t) - f(x) = O(|t|^\alpha)$ uniformly in (c, d), and $0 < \alpha < 1$, then $\mathcal{H}f \in \mathrm{Lip}_\alpha(c, d)$ also, but in general the Hilbert transform of a continuous function need not be continuous. The transform of a continuous function does, however, show 'traces of continuity', in that $\mathcal{H}f$ has the intermediate-value property in the set of points F for which it is finite ([14, p. 265]). These results for $\mathcal{H}f$ are usually proved for the *conjugate function*

$$\tilde{f}(x) = \lim_{\eta \to 0} \frac{1}{2\pi} \int\limits_{\eta \le |t-x| \le \pi} f(t) \cot \frac{x-t}{2}\, dt\,,$$

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in which context f is assumed to be a 2π-periodic function, but it is familiar that $\mathcal{H}f$ and $\tilde{f}$ have the same behaviour as a consequence of the fact that $1/x - \cot x$ has a continuous extension to a neighbourhood of 0.

In this note we consider the Hilbert transform of a function f in $L(\mathbb{R})$ which has bounded variation in a finite interval $[c, d]$, so that f has at most a countable set of points of discontinuity in $[c, d]$, and we describe certain continuity-type properties possessed by such transforms. More specifically, we show that, except for exceptional sets of $a \in (c, d)$ of capacity zero, $\mathcal{H}f(a)$ is finite and

$$\lim_{x \to a,\ x \notin E} \mathcal{H}f(x) \ = \ \mathcal{H}f(a), \tag{1.1}$$

where the excluded set E is metrically 'thin' at a, when measured in terms of an appropriate capacity. Results of this type, when applied to the conjugate function, lead to theorems involving the tangential boundary behaviour of analytic functions (see [6], [11], or [13], for example).

2. Capacity

The capacities we use to measure the size of exceptional sets are classical and involve the Bessel kernels G_α. An explicit integral formula for G_α, $0 < \alpha \le 1$, may be found in [2, p. 10], but for our purposes here it is enough to observe that G_α is an even, positive, and unbounded function in $L(\mathbb{R})$ which is decreasing in $(0, \infty)$, decays exponentially as $|x| \to \infty$, and satisfies

$$\begin{aligned} G_\alpha(x) \ &\simeq \ |x|^{\alpha-1}\,, \quad 0 < \alpha < 1\,, \\ G_1(x) \ &\simeq \ \log\frac{1}{|x|}\,, \end{aligned} \tag{2.1}$$

where $u \simeq v$ means that u/v is bounded above and below by positive constants for all sufficiently small non-zero $|x|$.

Definition 1. For a Borel set E and $0 < \alpha \le 1$, we define

$$C_\alpha(E) = \inf\{\,\mu(\mathbb{R})\},$$

where the infimum is taken over all non-negative Radon measures μ for which

$$\int_{\mathbb{R}} G_\alpha(x-t)\,d\mu(t) \ge 1 \quad \text{for all } x \in E.$$

Equivalently [5, p. 20],

$$C_\alpha(E) = \sup\{\,\mu(E)\},$$

where the supremum is taken over all measures $\mu \in \mathcal{M}^+(E)$, the class of non-negative Radon measures μ on $\mathbb{R}$ with support on E, for which

$$\int_{\mathbb{R}} G_\alpha(x-t)\,d\mu(t) \le 1 \quad \text{for all } x \in E.$$

We say that E has α-capacity zero if $C_\alpha(E) = 0$. Any set of α-capacity zero has Lebesgue measure zero, but the converse is false in general. Also, $C_\alpha(E) = 0$ implies $C_\beta(E) = 0$ for $0 < \beta < \alpha \le 1$. We shall use the term *logarithmic capacity* for the case $\alpha = 1$ of α-capacity. Logarithmic capacity is often defined differently in different contexts, but the definition given above, based on a classical approach, is suitable for the results under discussion here, and, as noted below, it yields a capacity that is comparable to a standard Bessel capacity from L^p-capacity theory [7].

For $0 < \alpha \le 1$, we shall say that a property that holds true for all $x \in (c,d) \setminus E$, where E has α-capacity zero, is true α-*quasi-everywhere* in the interval (c,d). For the case $\alpha = 1$ we shall usually simply write *quasi-everywhere*.

As an illustration of how sparse the points of a set of zero logarithmic capacity are, we note that if S is a Cantor set constructed in such a way that the set S_n obtained at the nth step consists of the union of 2^n disjoint intervals, each of length l_n, then $S = \bigcap_1^\infty S_n$ has zero logarithmic capacity if and only if $\sum_1^\infty 2^{-n} \log(1/l_n) = \infty$. (See [5, p. 31] or [2, Theorem 5.3.2].)

We derive next an elementary lower estimate for logarithmic capacity in terms of Lebesgue measure m. This lemma is a special case of a general result involving estimates of Bessel capacities in terms of Hausdorff measures (see [2, p. 139]), but we include a simple proof for the sake of completeness.

Lemma 1. *Suppose that $E \subset \mathbb{R}$ and that $0 < m(E) < 1/2$. Then*

$$\frac{A}{\log \frac{1}{m(E)}} \;\le\; C_1(E). \tag{2.2}$$

Here and below, A denotes a positive absolute constant, but not necessarily the same one at each occurrence.

Proof of Lemma 1. We define a measure $\mu \in \mathcal{M}^+(E)$ by setting $d\mu(t) = \beta\chi_E(t)dt$, where χ_E is the characteristic function of E and β is a positive constant that remains to be chosen. Then, for $x \in \mathbb{R}$, by the monotonicity of G_1,

$$\begin{aligned}\int_{\mathbb{R}} G_1(x-t)\,d\mu(t) &= \beta \int_E G_1(x-t)\,dt \\ &\le \beta \int_{-m(E)}^{m(E)} G_1(t)\,dt \\ &\le A\beta \int_{-m(E)}^{m(E)} \log\frac{1}{|t|}\,dt \;\le A\beta\, m(E)\log\frac{1}{m(E)}.\end{aligned}$$

If we now choose β so that the last quantity equals 1, then

$$\int_{\mathbb{R}} G_1(x-t)\,d\mu(t) \le 1 \quad \text{and} \quad \mu(E) = A/\log\frac{1}{m(E)}.$$

The required result follows from (the second part of) Definition 1. □

Remark. If $E = (-\delta, \delta)$ then $C_1(E) \simeq 1/\left(\log\frac{1}{\delta}\right)$ as $\delta \to 0$, see [2, p. 131].

3. Thin Sets and Fine Continuity

The notions of thin sets and fine continuity are generalisations of ideas from classical potential theory. (For this classical theory, see Armitage and Gardiner [1].) We base our definitions on those of Meyers [7] and Adams and Hedberg [2, Chapter 6]. These authors work with different capacities, particularly the Bessel capacities $C_{\alpha,p}$ ([7]), but noting that ([4, Corollary 2.2])

$$C_1(E) \;\le\; C_{1/2,2}(E) \;\le\; A\,C_1(E),$$

it is readily seen that the case $\alpha = 1$ of the following definition is a special case of the corresponding definitions in [8] and [2].

Definition 2. Suppose that $S \subset \mathbb{R}$ is a Borel set and that $0 < \alpha \le 1$. Then S is said to be *α-logarithmically thin at $a \in (c,d)$*, abbreviated to α-thin at a, if

$$\int_0^{t_0} C_1(S \cap (a-t, a+t))\,\frac{dt}{t^{2-\alpha}} < \infty \tag{3.1}$$

for some $t_0 > 0$. We say that a function $h : [c,d] \to \mathbb{R}$ is α-*finely continuous at* a if there is a set $S \subset \mathbb{R}$ such that S is α-thin at a and

$$\lim_{x\to a,\, x\in(c,d)\setminus S} h(x) = h(a).$$

Equivalently, see [2, Proposition 6.4.3], h is α-finely continuous at a if

$$\{x : x \in (c,d), |h(x) - h(a)| \geq \varepsilon\}$$

is α-thin at a for all $\varepsilon > 0$.

When $\alpha = 1$, we shall write *thin* and *finely continuous* for α-thin and α-finely continuous, respectively. We note that thin and finely continuous are then equivalent to the concepts of $(1/2, 2)$-thin and $(1/2, 2)$-finely continuous (with $N = 1$) in [2, Chapter 6].

If a set S is thin at a, then, for every fixed $\lambda > 1$,

$$m(S \cap (a-t, a+t)) \;=\; O(t^{\lambda}) \quad \text{as} \quad t \to 0. \tag{3.2}$$

To see that this is a consequence of (3.1) with $\alpha = 1$, note first that, for $t \in (0, t_0^2)$,

$$\frac{1}{2}\, C_1(S(t)) \,\log\frac{1}{t} = C_1(S(t)) \int_t^{t^{1/2}} \frac{dr}{r} \leq \int_t^{t^{1/2}} \frac{C_1(S(r))}{r}\, dr \;\to 0,$$

as $t \to 0$, where we have written $S(t)$ for $S \cap (a-t, a+t)$. Hence

$$C_1(S(t)) \;=\; o\left(\left(\log\frac{1}{t}\right)^{-1}\right) \quad \text{as} \quad t \to 0.$$

It is a simple matter to show, using (2.2), that (3.2) follows from this if $m(S(t)) > 0$ for $t > 0$. If $m(S(t_1)) = 0$ for some $t_1 > 0$, then $m(S(t)) = 0$ for $0 < t \leq t_1$, and (3.2) is trivially true. A similar argument to the above shows that (3.1), with $0 < \alpha < 1$, implies that $C_1(S(t)) = o(t^{1-\alpha})$ and hence, using (2.2) again, that

$$m(S \cap (a-t, a+t)) \;=\; O\left(\exp\bigl(-\frac{B}{t^{1-\alpha}}\bigr)\right) \quad \text{as} \quad t \to 0, \tag{3.3}$$

for every positive constant B.

4. Fine Continuity of the Hilbert Transform

Let $[c,d]$ be a finite closed interval and let $\chi \equiv \chi_{[c,d]}$ denote the characteristic function of $[c,d]$. Suppose $x \in (c,d)$ and set $\delta(x) = \min\{d-x,\, x-c\} > 0$. Then

$$\begin{aligned}\pi\mathcal{H}f(x) &= \lim_{\varepsilon\to 0}\int_{|x-t|>\varepsilon} \frac{f(t)\chi(t)}{x-t}\,dt + \int_{|x-t|>\delta(x)} \frac{f(t)(1-\chi(t))}{x-t}\,dt \\ &= \pi\mathcal{H}f\chi(x) + g(x)\end{aligned}$$

for $x \in (c,d)$. We show that g is continuous in (c,d). To this end, let $a \in (c,d)$ and write δ for $\delta(a)$. Then, if $t \in \mathbb{R}\setminus[c,d]$ and $|x-a| < \delta/2$, we have $|x-t| \ge \delta/2$ and

$$\left|\frac{f(t)(1-\chi(t))}{x-t}\right| \;\le\; \frac{2|f(t)|}{\delta}\,.$$

The continuity of g at a, and hence in (c,d), now follows from the Lebesgue dominated convergence theorem.

Suppose next that $f \in BV[c,d]$, i.e. that f has bounded variation on $[c,d]$. Then, by integration by parts, for $x \in (c,d)$ and $\varepsilon < \delta(x)$,

$$\begin{aligned}\int_{|x-t|>\epsilon} \frac{f(t)\chi(t)}{x-t}\,dt &= \int_{[c,d]\setminus(x-\varepsilon,x+\varepsilon)} \frac{f(t)}{x-t}\,dt \\ &= f(c)\log(x-c) - f(d)\log(d-x) \\ &\quad + \{f(x+\varepsilon) - f(x-\varepsilon)\}\log\varepsilon \\ &\quad + \int_{[c,d]\setminus(x-\varepsilon,x+\varepsilon)} \log|x-t|\,df(t).\end{aligned} \tag{4.1}$$

We now state a lemma on monotonic functions that we need.

Lemma 2. ([12, Lemma 1]; see also [10, Theorem VII]) *Suppose that F is increasing on $[c,d]$ and extended to $\mathbb{R}$ by setting $F(x) = F(d)$ for $x > d$ and $F(x) = F(c)$ for $x < c$. Then*

$$\int_c^d \log|x-t|\,dF(t) \tag{4.2}$$

is finite quasi-everywhere in $[c,d]$. If $x = a \in (c,d)$ is a value for which (4.2) *is finite, then*

$$F(a+\delta) - F(a-\delta) \;=\; o\left(1/\log\frac{1}{\delta}\right), \quad \delta\to 0, \tag{4.3}$$

and

$$\int_0^1 \frac{F(a+t)-F(a-t)}{t}\,dt \;<\; \infty\,. \tag{4.4}$$

Furthermore, for $0<\alpha<1$*, we have* ([12, p. 452])

$$\int_0^1 \frac{F(x+t)-F(x-t)}{t^{2-\alpha}}\,dt \;<\; \infty \tag{4.5}$$

for all $x\in(c,d)$*, except possibly for a set of* x *of* α*-capacity zero.*

Remark. An easy consequence of (4.5) and the monotonicity of F is that, for $0<\alpha<1$,

$$F(x+\delta)-F(x-\delta) \;=\; o(\delta^{1-\alpha}), \quad \delta\to 0, \tag{4.6}$$

α-quasi-everywhere in $[c,d]$. The relations (4.3) and (4.6), together with their associated exceptional sets, are intermediate results between two familiar facts for monotonic functions, namely that such functions are continuous outside a countable set and differentiable outside a set of Lebesgue measure zero.

Since $f\in BV[c,d]$, there are increasing functions F_1,F_2 for which $f=F_1-F_2$ on $[c,d]$. It thus follows from Lemma 2 and (4.1), since $C_1(E_1\cup E_2)=0$ if $C_1(E_1)=C_1(E_2)=0$, that

$$\begin{aligned}\pi\mathcal{H}f\chi(x) &= \lim_{\epsilon\to 0}\int_{|x-t|>\epsilon}\frac{f(t)\chi(t)}{x-t}\,dt\\ &= f(c)\log(x-c)-f(d)\log(d-x)+\int_c^d \log|x-t|\,df(t)\end{aligned} \tag{4.7}$$

is finite quasi-everywhere in (c,d).

We now state our main theorem.

Theorem 1. *If* $f\in L(\mathbb{R})$ *and* $f\in BV[c,d]$*, then* $\mathcal{H}f$ *is* α*-finely continuous* α*-quasi-everywhere in* $[c,d]$*, where* $0<\alpha\le 1$.

As noted above, $\mathcal{H}f=\mathcal{H}f\chi+g$, where g is continuous in (c,d), so, by (4.7), Theorem 1 will follow once we prove the following result.

Theorem 2. *Suppose that* F *is increasing on* $[c,d]$ *and extended to* $\mathbb{R}$ *as in Lemma 2. Set*

$$h(x)=\int_c^d \log\frac{1}{|x-t|}\,dF(t), \quad x\in[c,d]. \tag{4.8}$$

Suppose that $a \in (c,d)$ and that $h(a)$ is finite. Then h is finely continuous at a. Furthermore, for $0 < \alpha < 1$, the logarithmic potential h is α-finely continuous α-quasi-everywhere in $[c,d]$.

Remarks. 1. The case $\alpha = 1$ of Theorem 1 follows from the first part of Theorem 2 since, by Lemma 2, h is finite quasi-everywhere in $[c,d]$. This particular result concerning the logarithmic potential is known and is a consequence of standard results in potential theory (see [9] for an explicit formulation and an alternative approach to the one outlined here).

2. If h is finely continuous at a, then

$$\lim_{x \to a,\, x \notin E} h(x) \;=\; h(a),$$

where E is thin at a and consequently, by the remarks at the end of Section 3,

$$m(E \cap (a-\delta, a+\delta)) \;=\; O(\delta^{\lambda}), \quad \delta \to 0,$$

for every $\lambda > 1$. It follows that the result for the case $\alpha = 1$ in Theorem 1 here, when stated in terms of the conjugate function $\tilde{f}$ (see the Introduction above), sharpens the conclusion of [13, Theorem 1].

Proof of Theorem 2. We begin with the proof of the first part of the theorem. Suppose that $h(a)$ is finite where $a \in (c,d)$. Let $\varepsilon > 0$ be given and set

$$E(\varepsilon) \;=\; \{x \in [c,d] : |h(x) - h(a)| \geq \varepsilon\}.$$

It is enough, by (the second part of) Definition 2, to prove that $E(\varepsilon)$ is thin at a. We assume, as we may, that $d - c \leq 1$, so that the integrand in (4.8) is non-negative. By Fatou's lemma,

$$\begin{aligned}\liminf_{x \to a} h(x) \quad &= \quad \liminf_{x \to a} \int_c^d \log \frac{1}{|x-t|}\, dF(t) \\ &\geq \quad \int_c^d \liminf_{x \to a} \log \frac{1}{|x-t|}\, dF(t) \;=\; h(a),\end{aligned}$$

and it follows that

$$\liminf_{x \to a,\, x \in E(\varepsilon)} h(x) \;\geq\; h(a) + \varepsilon. \tag{4.9}$$

We show next that, for all sufficiently small ρ,

$$C_1(E(\varepsilon) \cap B(a,\rho)) \;\leq\; \frac{A}{\varepsilon}\left[F(a+2\rho) - F(a-2\rho)\right], \tag{4.10}$$

where $B(a,\rho) = (a-\rho, a+\rho)$ for $\rho > 0$. The short argument we use to do this is a simple adaptation of the argument used in [2, p. 180] to derive an analogous inequality for potentials of L^p functions.

We begin by setting

$$h_r(x) \;=\; \int_{t\in B(a,r)} \log\frac{1}{|x-t|}\,dF(t), \quad x\in[c,d],$$

where $r \in (0,1/2)$ is chosen so that $B(a,r) \subset [c,d]$ and

$$h_r(a) \;\le\; \frac{\varepsilon}{4}. \tag{4.11}$$

Such a choice of r is possible by the finiteness of $h(a)$. We note that, since

$$\lim_{x\to a}\int_{B'(a,r)} \log\frac{1}{|x-t|}\,dF(t) = \int_{B'(a,r)} \log\frac{1}{|a-t|}\,dF(t),$$

where $B'(a,r) = [c,d]\setminus B(a,r)$, it follows from (4.9) that

$$\liminf_{x\to a,\, x\in E(\varepsilon)} h_r(x) \ge h_r(a) + \varepsilon.$$

Let $0 < \rho \le r$. Then, if $|x-a| \le \rho/2$ and $|t-a| \ge \rho$, we have $|t-x| \ge |t-a|/2$, and so, using (4.11),

$$\begin{aligned}\int_{B(a,r)\setminus B(a,\rho)} \log\frac{1}{|x-t|}\,dF(t) &\le \int_{B(a,r)} \log\frac{2}{|a-t|}\,dF(t)\\ &\le 2\,h_r(a) \;\le\; \frac{\varepsilon}{2}.\end{aligned}$$

We now choose $\rho_0 \in (0,r)$ such that

$$h_r(x) \;\ge\; \frac{3\varepsilon}{4}$$

for $|x-a| \le \rho_0$ and $x \in E(\varepsilon)$. Then, for $0 < \rho < \rho_0$ and $x \in E(\varepsilon)\cap B(a,\frac{\rho}{2})$,

$$\int_{B(a,\rho)} \log\frac{1}{|x-t|}\,dF(t) = \int_{B(a,r)} - \int_{B(a,r)\setminus B(a,\rho)} \ge \frac{\varepsilon}{4}\,,$$

that is,

$$\int_{\mathbb{R}} \log\frac{1}{|x-t|}\,d\mu(t) \;\ge\; 1, \quad x\in E(\varepsilon)\cap B(a,\frac{\rho}{2}),$$

where $d\mu(t) = (4/\varepsilon)\chi_{B(a,\rho)}(t)\,dF(t)$. Since, by (2.1),

$$\log(1/|x-t|) \le A\,G_1(x-t)$$

for $x, t \in [c, d]$, it follows from (the first part of) Definition 1, that

$$C_1(E(\varepsilon) \cap B(a, \frac{\rho}{2})) \ \leq \ A\,\mu\,(B(a,\rho)) \leq \frac{4\,A}{\varepsilon}\,[F(a+\rho) - F(a-\rho)]\ ,$$

and we have established (4.10). From (4.4) we now easily obtain

$$\int_0^1 \frac{1}{t}\,C_1(E(\varepsilon) \cap B(a,t))\,dt \ < \ \infty,$$

so $E(\varepsilon)$ is thin at a. This proves the first part of Theorem 2. The case $0 < \alpha < 1$ follows from (4.10) and the last statement of Lemma 2. This completes the proof of Theorem 2 and hence the proof of Theorem 1 as well. $\square$

We conclude with an observation relating to Theorem 1. The exceptional set of $a \in (c, d)$ in Theorem 1 associated with a value of $\alpha \in (0, 1)$ is, in general, larger than the exceptional set corresponding to $\alpha = 1$, since $C_1(E) = 0$ implies $C_\alpha(E) = 0$ and the converse is false, but the excluded set E at a for which

$$\lim_{x \to a,\ x \notin E} \mathcal{H}f(x) \ = \ \mathcal{H}f(a)$$

is smaller, as indicated by a comparison between (3.2) and (3.3) with S replaced by E.

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