A GROWTH THEOREM FOR
BIOLOMORPHIC MAPPINGS
ON A BANACH SPACE

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Abstract. Let \( \| \cdot \| \) be an arbitrary norm on a Banach space \( E \). Let
\( B \) be the open unit ball of \( E \) for the norm \( \| \cdot \| \), and let \( f : B \to E \)
be a biholomorphic convex mapping such that \( f(0) = 0 \) and \( df(0) \) is
identity. We will give an upper bound of the growth of \( f \).

1. Introduction

Let \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \) denote the open unit disc in the complex
plane \( \mathbb{C} \). Let \( f : \Delta \to \mathbb{C} \) be a biholomorphic convex mapping with
\( f(0) = 0 \) and \( f'(0) = 1 \). Then the following inequality holds :
\[
|f(z)| \leq \frac{|z|}{1 - |z|}, \quad \text{for all } z \in \Delta.
\]

It is natural to consider a generalization of the growth theorem
above to \( \mathbb{C}^n \). Let \( \Omega \) be a domain in \( \mathbb{C}^n \) which contains the origin
in \( \mathbb{C}^n \). A holomorphic mapping \( f : \Omega \to \mathbb{C}^n \) is said to be
normalized, if \( f(0) = 0 \) and the Jacobian matrix \( Df(0) \) at the
origin is identity. Let \( B^n \) denote the Euclidean unit ball in \( \mathbb{C}^n \).

Let \( f : B^n \to \mathbb{C}^n \) be a normalized biholomorphic convex map-
and T. J. Suffridge [12] extended the upper bound above for the
growth of \( f \) to \( B^n \) in \( \mathbb{C}^n \) by using different methods and showed
that
\[
\|f(z)\|_2 \leq \frac{\|z\|_2}{1 - \|z\|_2} \quad \text{for all } z \in B^n,
\]

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where $\| \cdot \|_2$ denotes the Euclidean norm. Let

$$B_p = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \| z \|_p = (\sum_{i=1}^{n} |z_i|^p)^{1/p} < 1 \}$$

for $p \geq 1$ and

$$D(p_1, \ldots, p_n) = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^{p_1} + \cdots + |z_n|^{p_n} < 1 \}$$

with $p_1, \ldots, p_n \geq 1$. S. Gong and T. Liu [7] gave the upper bound above for the growth of normalized biholomorphic convex mappings on $B_p$ and $D(p_1, \ldots, p_n)$ and H. Hamada [8] proved a similar result on the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm. In this paper, we give the upper bound for the growth of a biholomorphic convex mapping on the unit ball in a complex Banach space as follows.

**Main Theorem** Let $E$ be a complex Banach space with norm $\| \cdot \|$. Let $B$ be the open unit ball of $E$ for the norm $\| \cdot \|$, and let $f : B \to E$ be a biholomorphic convex mapping such that $f(0) = 0$ and $df(0)$ is identity. Then

$$\| f(z) \| \leq \frac{\| z \|}{1 - \| z \|}$$

for all $z \in B$.

**2. Notation and preliminaries**

Let $U$ be an open set in a complex normed space $E$ and let $F$ be a complex Banach space. Let $f$ be a holomorphic mapping from $U$ to $F$. Then the following equation holds in a neighbourhood $V$ of $x$ in $U$ for $x \in U$:

$$f(z) = \sum_{n=1}^{\infty} P_n(z - x), \quad (2.1)$$

where

$$P_n(y) = \frac{d^n f(x)}{n!}(y) = \frac{1}{2\pi \sqrt{-1}} \int_{|\zeta|=1} \frac{1}{\zeta^{n+1}} f(x + \zeta y) d\zeta$$
for any \( y \in E \setminus \{0\} \) such that \( x + \zeta y \in U \) for all \( \zeta \in C \) with \( |\zeta| \leq 1 \). The series (2.1) is called the Taylor expansion of \( f \) by \( n \)-homogeneous polynomials \( P_n \) at \( x \).

Let \( E \) be a complex Banach space with norm \( \| \cdot \| \). Let \( B \) be the open unit ball of \( E \) for the norm \( \| \cdot \| \), and let \( f : B \to E \) be a biholomorphic mapping. A biholomorphic mapping \( f \) is said to be convex if \( f(B) \) is convex.

The following theorem (the Maximum Modulus Principle) is well-known (see, for example, Dunford and Schwartz [3]).

**Theorem 2.1** Let \( E \) be a complex Banach space with norm \( \| \cdot \| \). Let \( \Delta \) be the unit disc in \( C \), and let \( f : \Delta \to E \) be a holomorphic mapping. If there exists a point \( \zeta_0 \in \Delta \) such that \( \| f(\zeta_0) \| = \sup\{ \| f(\zeta) \| : \zeta \in \Delta \} \), then \( \| f(\zeta) \| \) is constant on \( \Delta \).

**3. Proof of Main Theorem**

Let \( \Delta \) be the unit disc in \( C \). We take a fixed boundary point \( w \in \partial B \). Let \( f(z) = \sum_{n=1}^{\infty} P_n(z) \) be the expansion of \( f \) by \( n \)-homogeneous polynomials \( P_n \) in a neighbourhood \( V \) of 0 in \( E \). Then we have \( f(z) = z + \sum_{n=2}^{\infty} P_n(z) \). For \( \zeta \in \Delta \),

\[
\left( 3.1 \right) \quad f(\zeta w) = \zeta w + \sum_{n=2}^{\infty} \zeta^n P_n(w).
\]

Let \( m \geq 2 \), \( m \in \mathbb{Z} \) be fixed. Let \( a = \exp(2\pi \sqrt{-1}/m) \). Then

\[
\sum_{k=0}^{m-1} f(\zeta^{\frac{1}{m}} a^k w) = \sum_{k=0}^{m-1} \{ \zeta^{\frac{1}{m}} a^k w + \sum_{n=2}^{\infty} (\zeta^{\frac{1}{m}} a^k)^n P_n(w) \}
\]

\[
= \zeta^{\frac{1}{m}} (\sum_{k=0}^{m-1} a^k w + \sum_{n=2}^{\infty} \sum_{k=0}^{m-1} a^{kn}) \zeta^{\frac{n}{m}} P_n(w)
\]

\[
= m \sum_{j=1}^{\infty} \zeta^j P_{jm}(w).
\]

This is holomorphic with respect to \( \zeta \in \Delta \). Since \( f(B) \) is convex, we have

\[
\frac{1}{m} \sum_{k=0}^{m-1} f(\zeta^{\frac{1}{m}} a^k w) \in f(B).
\]
We set
\[ h(\zeta) = f^{-1}(\frac{1}{m} \sum_{k=0}^{m-1} f(\frac{\zeta}{m} a^k w)). \]

Then \( h(\zeta) \) is holomorphic on \( \Delta \). By the conditions on \( f \),
\[ f^{-1}(z) = z + O(\|z\|^2). \]

We have
\[
\begin{align*}
  h(\zeta) &= f^{-1}\left( \sum_{j=1}^{\infty} \zeta^j P_m(w) \right) \\
  &= \sum_{j=1}^{\infty} \zeta^j P_m(w) + O(\| \sum_{j=1}^{\infty} \zeta^j P_m(w) \|^2) \\
  &= \zeta P_m(w) + O(\|\zeta\|^2).
\end{align*}
\]

Therefore \( \frac{h(\zeta)}{\zeta} \) is a holomorphic mapping from \( \Delta \) into \( E \). If \( \varepsilon > 0 \) is sufficiently small, \( \frac{h(\zeta)}{\zeta} \) is continuous and holomorphic on the set \( \{ \zeta : |\zeta| \leq 1 - \varepsilon \} \). Since \( h(\Delta) \subset B \), by Theorem 2.1, we obtain
\[
\left\| \frac{h(\zeta)}{\zeta} \right\| < \frac{1}{1 - \varepsilon}
\]
on \( \{ \zeta : |\zeta| \leq 1 - \varepsilon \} \). Letting \( \varepsilon \) tend to 0, we have
\[
\| P_m(w) \| = \left\| \frac{h(\zeta)}{\zeta} \right\|_{\zeta=0} \leq 1
\]
for all \( m \geq 2 \). Then we have

\[
\| f(\zeta w) \| \leq \| \zeta w \| + \sum_{n=2}^{\infty} \zeta^n P_n(w) \|
\]

\[
\leq |\zeta| + \sum_{n=2}^{\infty} |\zeta|^n
\]

\[
= \frac{|\zeta|}{1 - |\zeta|}
\]

\[
= \frac{\|\zeta w\|}{1 - \|\zeta w\|}
\]

and the proof of the Main Theorem is complete.

Let \( D \) be a bounded convex balanced domain in a complex Banach space \( E \). The Minkowski function \( N_D(z) \) of \( D \) is defined by

\[
N_D(z) = \inf\{\alpha > 0 : z \in \alpha D\}
\]

for all \( z \in E \). Then \( N_D(z) \) is a norm on \( E \). By the Main Theorem, we obtain the following corollary.

**Corollary** Let \( D \) be a bounded convex balanced domain in a complex Banach space \( E \). Let \( f : D \to E \) be a biholomorphic convex mapping such that \( f(0) = 0 \) and \( df(0) \) is identity. Then

\[
N_D(f(z)) \leq \frac{N_D(z)}{1 - N_D(z)}
\]

for all \( z \in D \).

**References**


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