TEN YEARS OF LOCAL MULTIPLIERS[1]

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The purpose of this talk was to review the development of the theory of local multipliers on C^*-algebras since their inception about ten years ago. At the same time it was intended to provide an overview on the contents of, and the main ideas in, the forthcoming book by Pere Ara (Universitat Autònoma de Barcelona) and the speaker [**5**].

Although the concept had made its appearance under the name 'essential multipliers' in papers by Elliott [**10**] and Pedersen [**17**] in the mid 1970's, the theory did not take off until the early 1990's when P. Ara and the speaker started their collaboration on the subject. The terms *local multipliers* and *local multiplier algebra* were introduced in [**11**] and [**12**]. Following the approach by Ara and Mathieu, the definition is made in three steps:

(i) One first defines the *symmetric algebra of quotients* $Q_s(A)$ of a given C^*-algebra A in the sense of Kharchenko, which is endowed with a natural involution and order structure;

(ii) Secondly, one identifies the *bounded elements* (in the sense of Handelman-Vidav) within $Q_s(A)$; these form a *-subalgebra $Q_b(A)$, called the *bounded symmetric algebra of quotients* of A, and, using the identity element as an order-unit, a natural C^*-norm on $Q_b(A)$ is introduced;

[1] This survey summarizes two talks delivered at the *Centre de Recerca Matemàtica*, Barcelona and the *Colloquium in Honour of Prof. Wittstock's 60th Birthday* at the Universität des Saarlandes, Saarbrücken in September and October 1999, respectively. The author is grateful for theses invitations and acknowledges with thanks the support in each case.

(iii) The pre-C^*-algebra $Q_b(A)$ is completed with respect to its natural norm to obtain $M_{\mathrm{loc}}(A)$, the *local multiplier algebra* of A.

At first glance, the route via the symmetric algebra of quotients seems to be an unnecessary diversion in defining $M_{\mathrm{loc}}(A)$ when compared with the more direct definition as a direct limit of the multiplier algebras of closed essential ideals of A, as it was, e.g., done in the papers by Elliott and Pedersen. It emerged, however, that it is precisely this 'diversion' relying on the purely algebraic construction which allows to exploit the intimate interconnection of the algebraic and the analytic concepts, making our new approach so fruitful. The relation between the two (equivalent) definitions becomes apparent once one realizes that the essentially defined double centralizers, that are used in defining the elements in $Q_s(A)$, are not only order-bounded if they lie in $Q_b(A)$, but bounded linear operators and hence can be extended to the closure of the ideal on which they are defined. Since closed ideals in a C^*-algebra are idempotent, this entails that every such bounded essentially defined double centralizer effectively gives rise to a multiplier of a closed essential ideal of A, providing the link to the purely C^*-algebraic approach.

The first breakthrough via this new approach was obtained in the paper [**1**] in which the centre of the local multiplier algebra could be described by a local version of the well-known Dauns-Hofmann Theorem. This result is vital for the definition of the *bounded central closure* ${}^cA$ of A, which is the C^*-algebra generated by A and the centre $Z(M_{\mathrm{loc}}(A))$ of the local multiplier algebra (if, for simplicity of our discussion here, we assume that A is unital). The local Dauns-Hofmann Theorem implies that $A \mapsto {}^cA$ is in fact a closure operation and allows the (now obvious) definition of *boundedly centrally closed* C^*-algebras.

It is this class of C^*-algebras that takes the role of von Neumann algebras in the applications of local multipliers to operator theory on C^*-algebras. Already in the articles [**10**] and [**17**], the motivation for $M_{\mathrm{loc}}(A)$ had been its usefulness in studying the question of innerness of automorphisms and derivations of C^*-algebras. A large part of our work (see [**5**]) is therefore concerned with exploiting the flexibility that one gains when working with *local* rather than *global* multipliers. In particular, it is possible to solve far more operator equations (cf. [**13**]) which arise very naturally in many questions concerning the structure of a number of classes of operators between C*-algebras, see also [**15**]. One of the many such instances is the computation of the *cb*-norm of elementary operators in terms of the central Haagerup tensor product [**3**]. In the same vein, the approach via *non-commutative functional analysis*, i.e., completely bounded operators, turned out to be expedient in determining the norm of an inner derivation on an arbitrary C*-algebra [**14**], [**5**]. The result states that, whenever δ_a is an inner derivation on a C*-algebra A (where a can be taken from the multiplier algebra $M(A)$ of A), there exists a local multiplier $b \in M_{\text{loc}}(A)$ such that $\delta_a = \delta_b$ and $\|\delta_b\| = 2\,\|b\|$. This extends the well-known result for von Neumann algebras [**19**] but it has the further advantage that, in the case of a unital simple C*-algebra A, it reproduces the correct statement one would expect, that is, $\|\delta_a\| = 2\operatorname{dist}(a, \mathbf{C}1)$, whereas the traditional approach via the universal enveloping von Neumann algebra would fail (unless the latter is a factor, a rather rare case).

The symmetric algebra of quotients $Q_s(A)$ can be thought of as an infinite-dimensional non-commutative analogue of the concept of the field of fractions of an integral domain (however, we even allow the existence of non-zero ideals which multiply to zero!). Its purpose is therefore quite the same as the purpose of the rationals with respect to the integers: we want to find solutions to (multiplicative) equations that do not exist in the domain we started with (here, the original C*-algebra). A further complication in our situation stems from the possibility that the solutions one obtains may not be bounded, i.e., may not be contained in the *bounded* symmetric algebra of quotients $Q_b(A)$. A general strategy to deal with this problem was termed *'beyond the universe and back'* for the following reason. The local multiplier algebra is deemed to be the 'universal enlargement' of the C*-algebra A in which we (hope to) find the solutions to our equations. In a first step, we may only obtain solutions that lie outside $M_{\text{loc}}(A)$, 'beyond the universe'. Additional arguments are thus needed—and fortunately often available—to modify the solutions in $Q_s(A)$ to 'bring them back' to the local multiplier algebra. This technique, first exploited in [**2**], is used over and over again in [**5**].

In the early 1960's, Herstein initiated a programme in which he asked for a structure theory of Lie mappings (i.e., Lie derivations and Lie homomorphisms) on associative rings. Progress on these questions was made by Martindale by introducing the so-called Martindale ring of quotients, a one-sided version of the symmetric ring of quotients. For a state-of-the-art account of these problems in the situation of prime rings, we refer to [**8**], and for semiprime rings to [**7**]. The structure results for Lie isomorphisms seemed to imply that, in the case of C^*-algebras, automatic continuity modulo the centre should be a consequence. This was recently verified in [**9**] and, with a much simpler proof, in [**5**]; the latter argument can in fact be adapted to cover the case of arbitrary Banach algebras [**6**]. This is of particular interest to us since the Lie structure is trivial on commutative C^*-algebras, hence Lie mappings genuinely fall into the realm of non-commutative functional analysis.

Having discussed some of the applications of local multipliers in operator theory, we took a glimpse into the future. Of course, there are a lot of open questions arising from our investigations in [**5**]. Some of them concern the structure of the local multiplier algebra itself, thus taking us back to a theme on which we worked at the beginnings. In [**4**], the first non-trivial examples of C^*-algebras which have a simple local multiplier algebra were obtained. From these a much further ranging problem arises: Under what conditions will $M_{\mathrm{loc}}(A)$ be simple? Given the rather intricate application of non-stable K-theory in [**4**], the answer to this question is expected to be not easy.

Among the open problems in operator theory on C^*-algebras connected with local multipliers is the longstanding question, first raised in [**17**], whether every derivation of a C^*-algebra becomes inner in the local multiplier algebra. Pedersen's positive answer for *separable* C^*-algebras is discussed in detail in [**5**], and there have been further special cases covered recently [**16**], [**18**]. However, only the answer in full generality would unify the classical results for von Neumann algebras by Kadison and Sakai and simple C^*-algebras by Sakai obtained in the 1960's and 1970's, respectively.

There seems to be enough work left over in this area for the next ten years.

References

[1] P. Ara and M. Mathieu, *A local version of the Dauns-Hofmann theorem*, Math. Z. **208** (1991), 349–353.

[2] P. Ara and M. Mathieu, *An application of local multipliers to centralizing mappings on C^*-algebras*, Quart. J. Math. Oxford (2) **44** (1993), 129–138.

[3] P. Ara and M. Mathieu, *On the central Haagerup tensor product*, Proc. Edinburgh Math. Soc. **37** (1994), 161–174.

[4] P. Ara and M. Mathieu, *A simple local multiplier algebra*, Math. Proc. Camb. Phil. Soc. **126** (1999), 555–564.

[5] P. Ara and M. Mathieu, *"Local multipliers of C^*-algebras"*, Springer-Verlag, London, 2000, to appear.

[6] B. Aupetit and M. Mathieu, *The continuity of Lie homomorphisms*, Stud. Math., to appear.

[7] R. Banning and M. Mathieu, *Commutativity preserving mappings on semiprime rings*, Commun. Algebra **125** (1997), 247–265.

[8] K. I. Beidar, W. S. Martindale and A. V. Mikhalev, *"Rings with generalized identities"*, Marcel Dekker, New York, 1996.

[9] M. I. Berenguer and A. R. Villena, *Continuity of Lie isomorphisms on Banach algebras*, Bull. London Math. Soc. **31** (1999) 6–10.

[10] G. A. Elliott, *Automorphisms determined by multipliers on ideals of a C^*-algebra*, J. Funct. Anal. **23** (1976), 1–10.

[11] M. Mathieu, *On C^*-algebras of quotients*, Semesterbericht Funktionalanalysis **16** (1989), 107–120.

[12] M. Mathieu, *The symmetric algebra of quotients of an ultraprime Banach algebra*, J. Austral. Math. Soc. (Series A) **50** (1991), 75–87.

[13] M. Mathieu, *How to solve an operator equation*, Publ. Mat. **36** (1992), 743–760.

[14] M. Mathieu, *The cb-norm of a derivation*, in: "Algebraic methods in operator theory", R. E. Curto and P. E. T. Jørgensen (Eds.), Birkhäuser, Basel, Boston, 1994, pp. 144–152.

[15] M. Mathieu, *Representation theorems for some classes of operators on C*-algebras*, Bull. Ir. Math. Soc. **41** (1998), 44–56.

[16] M. Mathieu, *Derivations implemented by local multipliers*, Proc. Amer. Math. Soc. **126** (1998), 1133–1138.

[17] G. K. Pedersen, *Approximating derivations on ideals of C*-algebras*, Invent. Math. **45** (1978), 299–305.

[18] D. W. B. Somerset, *The local multiplier algebra of a C*-algebra*, II, J. Funct. Anal., to appear.

[19] L. Zsido, *The norm of a derivation in a W*-algebra*, Proc. Amer. Math. Soc. **38** (1973), 147–150.

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