THREE PROBLEMS IN ONE

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1. Introduction

Sometime over the summer I received a postcard from Gordon Lessells (Deputy Leader), Kevin Hutchinson (Team Leader) and Pat McCarthy (Observer), who were with the Irish Team at the 1999 IMO in Bucharest, bearing the following cryptic message:

> Find the smallest constant C such that
>
> $$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{i=1}^{n} x_i \right)^4 .$$
>
> Can you find the two–line Chinese proof or the Irish solution?

When I got the card I realized that the problem posed was one of the six problems selected for the IMO, and was triply intrigued by it. Firstly, it didn't appear to fall easily by any of the standard methods that IMO participants would know, and therefore struck me as possessing an element of novelty and, as J. E. Littlewood has said, "the discovery of a new inequality is a cause for celebration." Secondly, because I had no idea what would pass for an acceptable "Chinese Solution." Thirdly, I wondered how I could possibly be expected to rediscover the "Irish Solution."

I suspected that Pat McCarthy–a former student of mine–might have had a hand in coming up with the "Irish Solution." But could I rediscover it? Did the link between Pat and myself provide a clue? Was the "Irish Solution" somehow "genetic"?

When I met him at the recent IMS meeting in TCD, Gordon elicited my response to these questions. I had to confess that I hadn't found any solution, but that I would like to think about it some more. During the meeting I found the second solution outlined below, and when I communicated it to himself and Kevin Hutchinson, who was also at the IMS meeting, Kevin informed that it was essentially the "Irish Solution," discovered by Pat and himself, who were unhappy with the inelegance of the proposed solutions discussed at the Jury Meetings, and were motivated to produce a nicer one.

Gordon later told me that a solution similar to the "Irish Solution" was also found by an Iranian student during the contest, and described the "Chinese solution" outlined below, which is based on an idea that a Chinese student used during the contest, and was elaborated later on by one of the coordinators.

Here's the formulation of the problem that the students were given on the first morning of the Olympiad:

Problem 1 *Let $n \geq 2$ be a fixed integer.*

1. *Find the least constant C such that*

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{i=1}^{n} x_i \right)^4 ,$$

for all non-negative real numbers x_1, x_2,..., x_n.

2. *Determine when equality occurs for this value of C.*

2. The "Chinese Solution"

Let

$$M = \sum_{i=1}^{n} x_i^2 .$$

Then it is clear that

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq M \sum_{1 \leq i < j \leq n} x_i x_j ,$$

with equality if, and only if, at worst all but two of the variables are zero.

Now, as is well known, $4ab \le (a+b)^2$ if a and b are real, with equality if, and only if, $a = b$. Hence, applying this with $a = M$, $b = 2\sum_{1\le i<j\le n} x_i x_j$ we see that

$$\begin{aligned} M \sum_{1\le i<j\le n} x_i x_j &\le \frac{1}{8}\left(\sum_{i=1}^n x_i^2 + 2\sum_{1\le i<j\le n} x_i x_j\right)^2 \\ &= \frac{1}{8}\left((\sum_{i=1}^n x_i)^2\right)^2 \\ &= \frac{1}{8}\left(\sum_{i=1}^n x_i\right)^4, \end{aligned}$$

with equality if, and only if,

$$\sum_{i=1}^n x_i^2 = 2\sum_{1\le i<j\le n} x_i x_j = \frac{1}{2}(\sum_{i=1}^n x_i)^2.$$

That is, if, and only if, two of the variables are equal and the rest are zero.

Thus, $C = 1/8$, and equality holds if, and only if, all of the variables are zero or at most two are non-zero and equal.

3. The "Irish Solution"

This uses the following easily established identities:

$$8ab(a^2+b^2) = (a+b)^4 - (a-b)^4, \tag{1}$$

and

$$\sum_{1\le i<j\le n} x_i x_j (x_i^2 + x_j^2) = \sum_{i=1}^n x_i \sum_{i=1}^n x_i^3 - \sum_{i=1}^n x_i^4. \tag{2}$$

The first of these tells us that $C \ge 1/8$ and that equality holds in case $n = 2$ if, and only if, the variables are equal.

The second identity suggests an alternative formulation of the problem, viz., find the smallest positive constant C such that

$$\sum_{i=1}^{n} x_i^3(1-x_i) \leq C$$

subject to the constraints that $0 \leq x_i$, $i = 1, 2, \ldots, n$, and

$$\sum_{i=1}^{n} x_i = 1.$$

As a first shot at solving this we observe that if $0 \leq x \leq 1$, then

$$\begin{aligned} x^2(1-x) &= 4(\frac{x}{2})(\frac{x}{2})(1-x) \\ &\leq 4\left(\frac{\frac{x}{2}+\frac{x}{2}+1-x}{3}\right)^3 \\ &= \frac{4}{27}, \end{aligned}$$

by the Arithmetic–Geometric Mean inequality, with equality holding if, and only if, $x = 2/3$. Hence,

$$\begin{aligned} \sum_{i=1}^{n} x_i^3(1-x_i) &\leq \max_{1\leq i\leq n} x_i^2(1-x_i)\sum_{i=1}^{n} x_i \\ &\leq \frac{4}{27}. \end{aligned}$$

This establishes that $C \leq 4/27$, but it is clear that the bound is crude and not attainable.

To refine this argument, we distinguish three cases: Case (A), in which we suppose all of the x_i belong to the interval $[0, 1/2)$; Case (B), in which we suppose that exactly one of the x_i belongs to $(1/2, 1]$, and Case (C), in which we suppose that precisely two of the x_i are equal to $1/2$. We proceed in the knowledge that these cases exhaust all possibilities.

Since $x(1-x) - y(1-y) = (x-y)(1-x-y)$ it is easy to see that $x(1-x)$ is strictly increasing on $[0,1/2]$. Hence, too, $x^2(1-x)$ is strictly increasing on $[0,1/2]$, so that if $0 \le x < 1/2$, then $x^2(1-x) < 1/8$, and so, in Case (A),

$$\begin{aligned}\sum_{i=1}^{n} x_i^3(1-x_i) &\le \max_{1\le i\le n} x_i^2(1-x_i)\sum_{i=1}^{n} x_i \\ &= \max_{1\le i\le n} x_i^2(1-x_i) \\ &< \frac{1}{8}.\end{aligned}$$

Suppose next that one of the x_i belongs to $(1/2,1]$. Denote this by a. The sum of the remaining $n-1$ variables (which we now relabel as y_1, y_2, …, y_{n-1}) is $1-a < 1/2$, so that each $y_i \in [0, 1-a] \subset [0,1/2)$. It follows that

$$y_i^2(1-y_i) \le (1-a)^2 a,\ i = 1, 2, \ldots, n-1.$$

Hence

$$\begin{aligned}\sum_{i=1}^{n} x_i^3(1-x_i) &= a^3(1-a) + \sum_{i=1}^{n-1} y_i^3(1-y_i) \\ &\le a^3(1-a) + \max_{1\le i\le n-1} y_i^2(1-y_i)\sum_{i=1}^{n-1} y_i \\ &\le a^3(1-a) + (1-a)^3 a \\ &= \frac{1}{8} - (2a-1)^4 \\ &< \frac{1}{8},\end{aligned}$$

by identity (1).

This disposes of Case (B); we are left with Case (C), which is trivial.

The upshot of this analysis is that, if $0 \leq x_i$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^n x_i = 1$, then

$$\sum_{i=1}^n x_i^3(1 - x_i) \leq \frac{1}{8},$$

and that the inequality is strict unless two of the variables are equal to $1/2$ and the rest are zero.

4. A Problem for the Reader

Let $n \geq 2$ be a positive integer and p a positive number. Determine (a) the best constant C_p such that

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^p + x_j^p) \leq C_p \left(\sum_{i=1}^n x_i \right)^{p+2},$$

for all nonnegative real numbers $x_1, x_2, \ldots, x_n$, and (b) the cases of equality.

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