# A SURVEY OF C*-ALGEBRAIC QUANTUM GROUPS, PART I

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**Abstract:** This is an overview of $C^*$-algebraic quantum groups. We begin with elementary Hopf algebra theory and define a finite quantum group as a Hopf $^*$-algebra that is a Frobenius algebra. The duality theory for finite quantum groups is thoroughly developed and includes a generalization of the Plancherel Formula.

Next we consider the more general case of compact quantum groups as defined by S. L. Woronowicz. We show how compact groups and duals of discrete groups fit into this category. The famous example quantum $SU(2)$ found by Woronowicz is then treated. It was the first example of a compact quantum group that is not a Kac algebra.

We develop the finite-dimensional co-representation theory and discuss the generalized Tannaka-Krein Theorem. The Haar functional (whose existence is one of the major achievements of Woronowicz's theory) is used to establish a Peter-Weyl type theorem for matrix elements of unitary co-representations. As opposed to the situation for Kac algebras the antipode is in general not involutive. This deviation is governed by a one-parameter group of automorphisms, which is also related to the fact that the Haar functional is not a trace.

The algebra of 'regular functions' of a compact quantum group is a Hopf $^*$-algebra. It is a dense subalgebra of the C$^*$-algebra occurring in the definition of the quantum group. The proof of these facts uses the left regular co-representation. It is an infinite-dimensional co-representation, which is a notion involving multiplier algebras. Compact quantum groups may be investigated via left regular co-representations, or rather multiplicative unitaries. We give an account of this approach due to S. Baaj and G. Skandalis and later modified by Woronowicz.

In the final part of this survey, to be published in the next issue of the *Bulletin*, we consider the category of multiplier Hopf $^*$-algebras

introduced by A. Van Daele, we show how $C^*$-algebraic quantum groups are related to the examples of quantum groups studied by V. G. Drinfeld and his collaborators, and we consider briefly the theory of general locally compact quantum groups, stating the recently established definition of a locally compact quantum group given by S. Vaas and the first author.

Throughout this paper we use the symbol $\odot$ to denote an algebraic tensor product and $\otimes$ to denote its topological completion with respect to the minimal tensor-product norm. The only exceptions are made when we discuss $h$-adic completions in Section 6 (part II).

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## 1. Introduction

The aim of this paper is to introduce the reader to the fascinating subject of quantum groups from the C$^*$-algebra point of view. Quantum groups were discovered a generation ago, and have since then developed in radically different directions motivated both from physics and mathematics. There exist several excellent treatments covering parts of the vast topic of quantum groups (see [14],[25]), but none seem to be seriously concerned with portraying in a broad manner the work done in the C$^*$-algebra framework. An explanation for this may be that important work on the general locally compact quantum group case has been accessible only in preprint form.

Loosely speaking, quantum groups are essentially groups or group-like objects that are quantizations of groups. We shall explain this in great detail later, but, for the moment, let us consider the situation more heuristically. Perhaps one of the most fruitful ideas in mathematics is to study geometrical spaces via naturally associated rings or algebras. A classical example is that of a compact Hausdorff space $X$ and the C*-algebra $C(X)$ of continuous functions on $X$. It is well known that all the topological information of the space $X$ is contained in the C*-algebraic structure of $C(X)$. In fact, Gelfand's theorem tells us that the functor $X \mapsto C(X)$ is an anti-equivalence from the category of compact Hausdorff spaces to the category of commutative unital C*-algebras.

With the discovery of quantum physics it soon became clear that non-commutative algebras could be used to explain geometry on the scale of atoms and molecules. Quantum physics could then be seen as a generalization of the theory of classical geometrical spaces to a theory of a suitable category of algebras, in such a way that the full subcategory of commutative algebras corresponds to the classical geometrical spaces. In this way non-commutative algebras can conceptually be thought of as 'sets of functions on quantum spaces'. The concept of quantization is more specific than explained here. It is intimately connected to Poisson manifolds and Poisson brackets measuring the deviation in the non-commutative product from the commutative pointwise product in terms of a deformation parameter (thought of as Planck's constant). Section 6 (part II) is devoted to explaining this.

Of course, the appropriate category of algebras studied depends on what properties of the space one is interested in. Thus, the algebras (given with pointwise operations) may consist of polynomials (algebraic geometry), complex-analytic functions (complex geometry), smooth functions (differential geometry), continuous functions (topology) or measurable functions (measure theory).

Quantum physics is in its nature a probabilistic theory. It was J. von Neumann who gave a rigorous mathematical foundation for quantum mechanics, using von Neumann algebras, which together with the theory of weights generalizes the classical theory of Borel integration. With their powerful structure theory, von Neumann algebras have proved successful in many areas of mathematics. In fact, the earliest attempt to give a generalization of Pontryagin's duality theorem for abelian locally compact groups to arbitrary locally compact groups used certain von Neumann algebras known as *Kac algebras* (see [8]).

Von Neumann algebras are C*-algebras of a special kind. Whereas von Neumann algebras generalize Borel measure theory, C*-algebras form, via Gelfand's theorem, natural generalizations of locally compact Hausdorff spaces. This suggests that locally compact quantum groups should be defined using C*-algebras. A leading proponent of this approach is the Polish physicist and mathematician S. L. Woronowicz, see [39]. His viewpoint is very much in the spirit of non-commutative geometry developed by A. Connes (see [4]), who uses non-commutative C*-algebras as a framework for powerful and useful notions of differential geometry on quantum spaces.

The C*-algebra structure takes care of the quantum spaces only as topological spaces. More structure has to be imposed to capture a possible group-like structure on these quantum spaces. After all, it is clear that topological groups may be homeomorphic without being isomorphic — there exist, for instance, two non-isomorphic groups consisting of eight elements. For finite groups the extra structure we are talking about is that of a *Hopf algebra*. Here the group multiplication is encoded in what is called a *co-multiplication*.

Suppose $G$ is a finite group and let $\mathbb{C}(G)$ denote the unital ${}^*$-algebra of complex valued functions on $G$. The co-multiplication $\Delta$ is the unital ${}^*$-homomorphism from $\mathbb{C}(G)$ to the algebraic tensor product $\mathbb{C}(G) \odot \mathbb{C}(G)$ defined by $\Delta(f)(s,t) = f(st)$ for all $f \in \mathbb{C}(G)$ and $s, t \in G$. Here we have identified $\mathbb{C}(G) \odot \mathbb{C}(G)$ with the unital ${}^*$-algebra $\mathbb{C}(G \times G)$ of complex valued functions on $G \times G$. Thus $\Delta$ is simply the transpose of the multiplication of $G$ by the contravariant functor $G \mapsto \mathbb{C}(G)$ from the category of finite groups to the category of finite-dimensional commutative C*-algebras. The associativity of the group multiplication gives the identity $(\Delta \odot \iota)\Delta = (\iota \odot \Delta)\Delta$, known as *co-associativity*. In this way the functor, $G \mapsto \mathbb{C}(G)$, is used to transfer systematically group notions (such as the existence of the unit element and the inverse, together with their axioms) to notions about algebras (such as the existence of the *co-unit* and the *co-inverse* with corresponding identities). In the category of algebras these notions makes sense even for algebras that are not commutative and one arrives at the more general concept of a Hopf algebra. As we shall see in Section 2 , finite quantum groups are Hopf algebras with a good $^*$-operation. In Section 2 we develop the theory of such quantum groups to familiarize the reader with the functor, $G \mapsto \mathbb{C}(G)$, and the theory of Hopf algebras.

For finite groups the topology involved is the discrete one, and the generalization to finite quantum groups runs as smoothly as one could hope. For compact groups one considers instead the functor, $G \mapsto C(G)$, going into the category of unital C$^*$-algebras, where $C(G)$ is the algebra of continuous functions on the compact group $G$. One may define $\Delta$ using the same formula as before. In general $\Delta(C(G)) \subset C(G \times G)$ is not contained in the algebraic tensor product $C(G) \odot C(G)$. But $C(G \times G)$ may be identified with the topological completion $C(G) \otimes C(G)$ of $C(G) \odot C(G)$ with respect to a natural C$^*$-norm, so one arrives at a topological version of Hopf algebras.

There is no problem defining the co-unit and co-inverse on $C(G)$. In the mid-eighties a whole new class of group-like objects were discovered, and it became evident from these important examples — which clearly deserved to be called quantum groups — that the unit and co-inverse could not in general be defined as bounded operators on the non-commutative C$^*$-algebras involved. They also ruled out Kac algebras as determining a too restrictive class of quantum groups, exactly for the reason that the co-unit and the co-inverse were treated as bounded operators (on the von Neumann algebras involved).

It was Woronowicz who proposed the first definition (see [37]) of a compact quantum group general enough to contain his newly-discovered quantum group, a twisted version $SU_q(2)$ (see [38]), of the classical matrix group $SU(2)$. He proved the existence of the *Haar state* for such quantum groups and used it to extend the classical Peter-Weyl Theorem to the category of compact quantum groups that he called *compact matrix pseudo-groups*. Soon after, he proved a generalization of the celebrated Tannaka-Krein Theorem, which made it clear that the theory of compact quantum groups was in essence a theory of finite dimensional unitary co-representations. The matrix elements of these co-representations form a dense unital $^*$-algebra of the C$^*$-algebra of the quantum group, generalizing the algebra of regular functions on classical groups. The co-unit and co-inverse can be defined most naturally on this algebra. We outline Woronowicz's theory of compact quantum groups in Section 3 and Section 4.

In a subsequent paper [22] Woronowicz defined the dual of a compact quantum group, thereby generalizing classical discrete groups. For a discrete group $G$ it is natural to consider the functor $G \mapsto C_0(G)$, where $C_0(G)$ is the non-unital C$^*$-algebra of (continuous) functions on the discrete group $G$ vanishing at infinity. The formula for the co-multiplication $\Delta$ given above does not necessarily have range contained in $C_0(G \times G)$; rather, it belongs to the unital C$^*$-algebra $C_b(G \times G)$ of bounded functions on $G$, which may be identified with the multiplier algebra of $C_0(G) \otimes C_0(G)$. Thus one is led to the notion of *multiplier Hopf algebras*.

That the topology does not play a major role in the theory of compact and discrete quantum groups became evident from A. Van Daele's definition of algebraic quantum groups [29]. It is a purely algebraic definition and the category thus defined contains Woronowicz's compact and discrete quantum groups. Also Van Daele proved a generalization of Pontryagin's duality theorem within this category. The important notion is that of the Haar functional corresponding to the Haar integral for classical groups. It is used to define the Fourier transform which identifies the convolution algebra of the quantum group with the algebra of functions on its dual quantum group. In Section 5 (part II) we outline the theory of multiplier Hopf algebras and identify the compact and discrete quantum groups of Woronowicz.

For a compact Lie group there are other algebras naturally attached to it, namely its associated Lie algebra and the universal enveloping algebra of the Lie algebra consisting of left-invariant differential operators on the Lie group. This can be embedded as a unital Hopf $^*$-algebra into the maximal dual of the Hopf $^*$-algebra of regular functions on the Lie group. Unlike the case of the convolution algebra (which may also be seen as an algebra of linear functionals on the algebra of regular functions), these linear functionals are of course not bounded, so the C*-algebras come in very indirectly as algebras to which these differential operators are affiliated. Most of the interesting new examples of quantum groups where found as deformations or quantizations of these co-commutative Hopf algebras in the monumental work by V. G. Drinfeld and his collaborators. Section 6 (part II) is devoted to explain some of their work and how it is related to Woronowicz's theory.

In Section 7 (part II), we give a recent definition of locally compact quantum groups proposed by the first author and S. Vaas (see [16]). It is well known that the unitary representation theory of non-compact locally compact groups is highly nontrivial. For the matrix group $SL(2,\mathbb{R})$, for instance, there are no finite-dimensional unitary representations. The role of topology becomes much more significant in treating non-compact locally compact quantum groups, and the left and right invariant Haar weights play a vital role.

## 2. Hopf *-algebras, Finite Quantum Groups and Duality

We reformulate the theory of finite groups in terms of finite quantum groups by considering algebras of functions naturally attached to the groups. These algebras carry additional structures leading to the notion of a Hopf *-algebra. Hopf algebras have been subject to intensive studies by algebraists over the last decades, [1]. A finite quantum group is a finite-dimensional Hopf *-algebra which in addition is a C*-algebra or Frobenius algebra. In the category of finite quantum groups thus defined, the finite groups are identified as the full subcategories of commutative or co-commutative Hopf *-algebras, and the finite abelian groups as the Hopf *-algebras which are both commutative and co-commutative. In this section we establish a duality result within the category of finite quantum groups that generalizes Pontryagin's duality theorem for finite abelian groups.

The crucial role played by Haar functionals on the Hopf *-algebras involved, corresponding to Haar integrals for groups, will be evident. Among the important identities obtained is the generalization of Plancherel's formula for finite abelian groups. It turns out that the duality theory developed here (which uses the Haar functionals so manifestly) has found its formulation in the broad framework of multiplier Hopf $^*$-algebras developed by A. Van Daele (see Section 5 (part II)). They contain all compact and discrete groups.

Historically the function algebra $K(G)$, introduced below, led to the theory of compact quantum groups developed by Woronowicz in the C$^*$-algebra context (see [31] and [37]). The convolution algebra $\mathbb{C}[G]$, also defined below, suggested the approach to quantum groups in terms of Kac algebras [8]. Later we shall study a third approach by considering the universal enveloping algebras of Lie algebras, which are again Hopf $^*$-algebras, but now consisting of unbounded elements. This approach is emphasized by the Russian school and includes the monumental work by Drinfeld, [7].

Here we have chosen to stick to the finite-dimensional case due to the technical difficulties otherwise encountered, and we hope that our rather detailed account, including proofs, will reveal the essential ideas necessary to resolve problems arising in the development of the more general theory.

The first example, and the most relevant to our approach, comes with the function algebra $K(G)$ of a group $G$. To simplify matters, we shall assume $G$ to be finite. Later, locally compact (infinite) groups will enter the arena, and this approach proposes C$^*$-algebras as a framework for locally compact quantum groups.

Let $K(G)$ be the set of all complex-valued functions on $G$. It is a unital $^*$-algebra under the following operations:

- $(\lambda f)(s) = \lambda\, f(s)$,
- $(f + g)(s) = f(s) + g(s)$,
- $(fg)(s) = f(s)g(s)$,
- $f^*(s) = \overline{f(s)}$,
- $1(s) = 1$,

where $f, g \in K(G)$, $\lambda \in \mathbb{C}$ and $s \in G$. Throughout this paper we use the symbol $\odot$ to denote the algebraic tensor product of two vector spaces. Since $G$ is finite, we may identify the algebras

$K(G) \odot K(G)$ and $K(G \times G)$ via the formula $(f \otimes g)(s,t) = f(s)\,g(t)$ for all $f, g \in K(G)$ and $s, t \in G$. We transpose the group structure to $K(G)$ by introducing:

• A unital $^*$-homomorphism $\Delta : K(G) \to K(G) \odot K(G)$ such that $\Delta(f)(s,t) = f(st)$.
• A unital $^*$-homomorphism $\varepsilon : K(G) \to \mathbb{C}$ such that $\varepsilon(f) = f(e)$.
• A unital involutive $^*$-automorphism $S : K(G) \to K(G)$ such that $S(f)(s) = f(s^{-1})$.

Here $f \in K(G)$ and $s, t \in G$, whereas $e$ denotes the neutral element in $G$.

The group axioms can be expressed in terms of these maps by the following identities:

1. $(\Delta \odot \iota)\Delta = (\iota \odot \Delta)\Delta$,
2. $(\varepsilon \odot \iota)\Delta = (\iota \odot \varepsilon)\Delta = \iota$,
3. $m(S \odot \iota)\Delta = m(\iota \odot S)\Delta = 1\,\varepsilon(.)$,

where $\iota$ is the identity map on $K(G)$ and $m : K(G) \odot K(G) \to K(G)$ is the multiplication on $K(G)$ lifted to the tensor product $K(G) \odot K(G)$, so $m(x \otimes y) = xy$ for $x, y \in K(G)$. The first identity is a consequence of the associativity of the group multiplication. The second identity expresses the fact that $e$ is the neutral element of $G$ and the last one stems from the axiom for inverse elements.

What we have at hand is an example of a commutative Hopf $^*$-algebra. Let us recall the general definition:

**Definition 2.1** *Consider a unital $^*$-algebra $A$ (with multiplication $m : A \odot A \to A : a \otimes b \mapsto ab$ ) and a unital $^*$-homomorphism $\Delta : A \to A \odot A$ satisfying co-associativity, i.e. $(\Delta \odot \iota)\Delta = (\iota \odot \Delta)\Delta$. Assume furthermore the existence of linear maps $\varepsilon : A \to \mathbb{C}$ and $S : A \to A$ fulfilling the conditions:*

$$(\varepsilon \odot \iota)\Delta = (\iota \odot \varepsilon)\Delta = \iota, \qquad (2.1.1)$$

$$m(S \odot \iota)\Delta = m(\iota \odot S)\Delta = 1\,\varepsilon(.)\ . \qquad (2.1.2)$$

*The pair $(A, \Delta)$ is called a Hopf $^*$-algebra.*

The linear maps $\varepsilon$ and $S$ are uniquely determined by the conditions (1) and (2) (see [1]). They are called *co-unit* and *antipode*, respectively, whereas the term *co-multiplication* is used for $\Delta$. We collect some basic properties:

**Proposition 2.2** *Consider a Hopf* $^*$*-algebra* $(A, \Delta)$ *with co-unit* $\varepsilon$ *and antipode* $S$. *Then*

1. $\varepsilon$ *is a unital* $^*$*-homomorphism and* $\varepsilon S = \varepsilon$.
2. $S$ *is anti-multiplicative and* $S(S(a^*)^*) = a$ *for all* $a \in A$.
3. $\sigma(S \odot S)\Delta = \Delta S$, *where* $\sigma : A \odot A \to A \odot A$ *is the flip-automorphism given by* $\sigma(a \otimes b) = b \otimes a$ *for all* $a, b \in A$.

We say that two Hopf $^*$-algebras $(A_1, \Delta_1)$ and $(A_2, \Delta_2)$ are *isomorphic* if there exists a $^*$-isomorphism $\pi : A_1 \to A_2$ such that $(\pi \odot \pi)\Delta_1 = \Delta_2\pi$. Notice that the uniqueness of the co-unit and the antipode implies that $\varepsilon_2\pi = \varepsilon_1$ and $S_2\pi = \pi S_1$.

The two linear maps $T_1, T_2 : A \odot A \to A \odot A$ determined by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \qquad \text{and} \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b)\ ,$$

where $a, b \in A$, play a fundamental role in the theory of quantum groups. They are linear isomorphisms with inverses given by

$$T_1^{-1}(a\otimes b) = ((\iota\odot S)\Delta(a))\,(1\otimes b) \quad \text{and} \quad T_2^{-1}(a\otimes b) = (a\otimes 1)\,((S\odot\iota)\Delta(b))$$

for $a, b \in A$.

Returning to our example $(K(G), \Delta)$, the formulas in Proposition 2.2 are seen to be dual versions of the well known identities $e^{-1} = e$, $(gh)^{-1} = h^{-1}g^{-1}$ and $(g^{-1})^{-1} = g$ in the group $G$. The maps $T_1$ and $T_2$ are transposes of the bijective maps

$$G \times G \to G \times G : (s, t) \mapsto (st, t) \quad \text{and} \quad G \times G \to G \times G : (s, t) \mapsto (s, st),$$

respectively, where $s, t \in G$. The bijectivity of $T_1$ and $T_2$ expresses the fact that the maps $s \mapsto st$ and $s \mapsto ts$ from $G$ to $G$, where $s, t \in G$, are bijective. Since $G$ is finite it means that each of them is either surjective or injective.

It is easy to prove that the set of characters (that is, unital $^*$-homomorphisms) on $K(G)$ is a group isomorphic to $G$ under multiplication given by $\lambda\eta = (\lambda \otimes \eta)\Delta$ for characters $\lambda, \eta$ of $K(G)$. Thus if $G_1$ and $G_2$ are finite groups, then they are isomorphic if and only if the associated Hopf $^*$-algebras $(K(G_1), \Delta_1)$ and $(K(G_2), \Delta_2)$ are isomorphic.

Furthermore, for a finite group $G$, observe that $K(G)$ is a finite-dimensional commutative C$^*$-algebra with respect to the supremum norm. Applying Gelfand's theorem, it is not hard to see that every Hopf $^*$-algebra $(A, \Delta)$, with $A$ a finite-dimensional commutative C$^*$-algebra, is isomorphic to $(K(G), \Delta)$ for some finite group $G$. We thus arrive at the following definition of a finite quantum group.

**Definition 2.3** *A finite quantum group $(A, \Delta)$ is a Hopf $^*$-algebra for which $A$ is a finite-dimensional Frobenius algebra, i.e. $a^*a = 0 \Leftrightarrow a = 0$ for all $a \in A$.*

A linear functional $h$ on $A$ is called a *Haar functional* on a Hopf $^*$-algebra $(A, \Delta)$ if $h(1) \neq 0$ and $(h \odot \iota)\Delta = 1\, h(\cdot)$.

Assume that $h_1$,$h_2$ are linear functionals on $A$ such that $h_1(1) = h_2(1) \neq 0$ and $(h_1 \odot \iota)\Delta = 1\, h_1(\cdot)$ and $(\iota \odot h_2)\Delta = 1\, h_2(\cdot)$. Then, for all $a \in A$, we have $h_2(1)\, h_1(a) = h_2\big(h_1(a)\, 1\big) = h_2\big(\, (h_1 \odot \iota)\Delta(a)\,\big) = h_1\big(\, (\iota \odot h_2)\Delta(a)\,\big) = h_1(1)\, h_2(a)$, and hence $h_1 = h_2$.

If follows from statement 3 in Proposition 2.2 that a Haar functional $h$ on $(A, \Delta)$ satisfies $(\iota \odot hS)\Delta = 1\,(hS)(\cdot)$ and $(hS)(1) = h(1)$. Thus $h = hS$ and $(\iota \odot h)\Delta = 1\, h(\cdot)$ and $h$ is unique up to multiplication by a scalar.

Any finite quantum group $(A, \Delta)$ possesses a unique (up to a scalar) positive Haar functional $h$, i.e. a Haar functional such that $h(a^*a) \geq 0$ for all $a \in A$. Moreover, $h$ is faithful. We discuss this in Section 3 in the more general context of compact quantum groups.

These results are evident for $A$ commutative: Suppose $G$ is a finite group and define a linear functional $h : K(G) \to \mathbb{C}$ by $h(f) = \sum_{s \in G} f(s)$ for all $f \in K(G)$. It is of course the integral corresponding to the counting measure on $G$.

Clearly $(h \odot \iota)(F)(t) = \sum_{s \in G} F(s,t)$ for all $F \in K(G \times G)$ and $t \in G$. Therefore

$$(h \odot \iota)(\Delta(f))(t) = \sum_{s \in G} \Delta(f)(s,t) = \sum_{s \in G} f(st) = \sum_{s \in G} f(s) = h(f)\, 1(t)$$

for all $t \in G$ and $f \in K(G)$. So $h$ is the Haar functional on $(K(G), \Delta)$. Clearly it is positive and faithful.

Given a discrete group $G$ (not necessarily finite), we construct a Hopf $^*$-algebra $(\mathbb{C}[G], \hat{\Delta})$ from $G$, which as we shall see later, is dual to the Hopf $^*$-algebra $(K(G), \Delta)$.

As a set, $\mathbb{C}[G]$ consists of the complex valued functions with finite support. It is a unital $^*$-algebra under the following operations:

- $(\lambda f)(s) = \lambda\, f(s)$,
- $(f+g)(s) = f(s) + g(s)$,
- $(fg)(s) = \sum_{t \in G} f(t)\, g(t^{-1}s)$,
- $f^*(s) = \overline{f(s^{-1})}$,

where $f, g \in \mathbb{C}[G]$, $\lambda \in \mathbb{C}$ and $s \in G$. Notice that the sum above is finite because $f$ has finite support.

For $s \in G$, define $\delta_s \in \mathbb{C}[G]$ to be equal to 1 at the point $s$ and equal to 0 elsewhere. Obviously $(\delta_s)_{s \in G}$ forms a basis for the vector space $\mathbb{C}[G]$. The formulas $\delta_s \delta_t = \delta_{st}$ and $(\delta_s)^* = \delta_{s^{-1}}$ for $s, t \in G$ will be useful in the sequel. For instance, it is immediate from them that $\delta_e$ is the unit element in $\mathbb{C}[G]$. Using the map $G \to \mathbb{C}[G] : s \mapsto \delta_s$, we can regard $G$ as a subgroup of the group of invertible elements in $\mathbb{C}[G]$.

We define linear maps specified on the basis $(\delta_s)_{s \in G}$:

- $\hat{\Delta}\ : \mathbb{C}[G] \to \mathbb{C}[G] \odot \mathbb{C}[G]$ by $\hat{\Delta}(\delta_s) = \delta_s \otimes \delta_s$,
- $\hat{\varepsilon}\ : \mathbb{C}[G] \to \mathbb{C}$ by $\hat{\varepsilon}(\delta_s) = 1$,
- $\hat{S}\ : \mathbb{C}[G] \to \mathbb{C}[G]$ by $\hat{S}(\delta_s) = \delta_{s^{-1}}$,

where $s \in G$. It follows that $(\mathbb{C}[G], \hat{\Delta})$ is a Hopf $^*$-algebra with co-unit $\hat{\varepsilon}$ and antipode $\hat{S}$:

The co-associativity expresses the associativity of the tensor product $\otimes$, i.e.

$$(\hat{\Delta} \odot \iota)\hat{\Delta}(\delta_s) = (\delta_s \otimes \delta_s) \otimes \delta_s = \delta_s \otimes (\delta_s \otimes \delta_s) = (\iota \odot \hat{\Delta})\hat{\Delta}(\delta_s),$$

where $s \in G$.

Denoting the multiplication map $\mathbb{C}[G] \odot \mathbb{C}[G] \to \mathbb{C}[G]$ by $\hat{m}$, we have $\hat{S}(\delta_s)\, \delta_s = \delta_{s^{-1}}\, \delta_s = \delta_{s^{-1}s} = \delta_e$ for all $s \in G$, and hence

$$\hat{m}(\hat{S} \odot \iota)\hat{\Delta}(\delta_s) = \hat{S}(\delta_s)\, \delta_s = \delta_e = \delta_e\, \hat{\varepsilon}(\delta_s) = \hat{m}(\iota \odot \hat{S})\hat{\Delta}(\delta_s)\,.$$

The identity (2.1.1) for the co-unit is proved in a similar fashion. It should be noted that the group multiplication is encoded in the convolution product $\hat{m}$ this time, and not in the co-multiplication, as is the case for $(K(G), \Delta)$.

Note again that $\hat{\Delta}(\delta_s) = \delta_s \otimes \delta_s$ for every $s \in G$. Suppose $(A, \Delta)$ is a Hopf $^*$-algebra. An element $a \in A$ is called *group-like* if $a \neq 0$ and $\hat{\Delta}(a) = a \otimes a$. This notion is justified by the fact that every group-like element in $(\mathbb{C}[G], \hat{\Delta})$ is of the form $\delta_s$ for some $s \in G$:

Suppose $f \in \mathbb{C}[G]$ is group-like and write $f = \sum_{s \in G} f(s)\, \delta_s$. Then

$$\sum_{s \in G} f(s)\, \delta_s \otimes \delta_s = \hat{\Delta}(f) = f \otimes f = \sum_{s,t \in G} f(s)\, f(t)\ \delta_s \otimes \delta_t\ .$$

Since $(\delta_s \otimes \delta_t)_{s,t \in G}$ is a basis for $\mathbb{C}[G] \odot \mathbb{C}[G]$, it follows that $f(s)\, \delta_s = f(s)\, f$ for all $s \in G$. As $f \neq 0$, there exists $r \in G$ such that $f(r) \neq 0$ and thus $f = \delta_r$.

Define a linear functional $\hat{h} : \mathbb{C}[G] \to \mathbb{C}$ by $\hat{h}(f) = f(e)$ for $f \in \mathbb{C}[G]$. Hence, $\hat{h}(\delta_e) = 1$ and $\hat{h}(\delta_s) = 0$ for all $s \in G \setminus \{e\}$. Because

$$(\hat{h} \odot \iota)\hat{\Delta}(\delta_s) = \hat{h}(\delta_s)\, \delta_s = \delta_e \hat{h}(\delta_s)$$

for all $s \in G$, we conclude that $\hat{h}$ is the Haar functional on $(\mathbb{C}[G], \hat{\Delta})$. Now

$$\hat{h}(f^* f) = (f^* f)(e) = \sum_{s \in G} |f(s)|^2$$

for all $f \in \mathbb{C}[G]$, so $\hat{h}$ is positive and faithful.

In the rest of this section we assume that $G$ is a finite group. By faithfulness of the Haar functional $\hat{h}$, we conclude that $(\mathbb{C}[G], \hat{\Delta})$ is a finite quantum group in the sense of Definition 2.3.

Notice that $(\mathbb{C}[G], \hat{\Delta})$ is a co-commutative Hopf $^*$-algebra, i.e. $\sigma\hat{\Delta} = \hat{\Delta}$, where $\sigma$ denotes the flip-automorphism. In fact, every finite co-commutative quantum group is isomorphic to

$(\mathbb{C}[G], \hat{\Delta})$ for some finite group $G$. This assertion is evident from the duality result below.

Let $(A, \Delta)$ be a finite quantum group with a positive faithful Haar functional $h$. The set of group-like elements of $(A, \Delta)$ forms a subgroup of the group of unitaries: Let $a, b \in A$ be group-like. Then $\varepsilon(a) = 1$ and $a^{-1} = S(a)$. Hence $ab \neq 0$ and thus $ab$ is group-like. Also $h(a)a = (h \otimes \iota)\Delta(a) = h(a)I$, and therefore $h(a) = 0$ unless $a = I$. Since $a^*$ is group-like, $a^*a$ is group-like. As $h(a^*a) > 0$, we thus get $a^*a = I$. Similarly $aa^* = I$ and so $S(a) = a^*$.

Furthermore, we shall see in Section 3 that the set of group-like elements of $(A, \Delta)$ forms a linear basis if and only if $(A, \Delta)$ is co-commutative.

We begin by defining the dual Hopf $^*$-algebra $(\hat{A}, \hat{\Delta})$ of a finite-dimensional Hopf $^*$-algebra $(A, \Delta)$ with co-unit $\varepsilon$ and antipode $S$. Let $\hat{A}$ be the vector space of linear functionals on $A$. We use the co-multiplication, the antipode and the $^*$-operation on $A$ to define a multiplication and $^*$-operation on $\hat{A}$:

- $(\omega\theta)(a) = (\omega \odot \theta)\Delta(a)$,
- $\omega^*(a) = \overline{\omega(S(a)^*)}$

for all $\omega, \theta \in \hat{A}$ and $a \in A$. Equipped with these operations $\hat{A}$ acquires the status of a unital $^*$-algebra with unit $\varepsilon$.

Since $A$ is finite-dimensional, we may identify $\hat{A} \odot \hat{A}$ with the vector space of linear functionals on $A \odot A$ via the formula $(\omega \otimes \theta)(a \otimes b) = \omega(a)\theta(b)$, where $\omega, \theta \in \hat{A}$ and $a, b \in A$.

Using this identification, the multiplication on $A$ yields a co-multiplication $\hat{\Delta} : \hat{A} \to \hat{A} \odot \hat{A}$ on $\hat{A}$ such that $\hat{\Delta}(\omega)(a \otimes b) = \omega(ab)$ for all $\omega \in \hat{A}$ and $a, b \in A$.

Similarly the unit 1 and the antipode $S$ on $(A, \Delta)$ define a co-unit $\hat{\varepsilon} : A \to \mathbb{C}$ and an antipode $\hat{S} : \hat{A} \to \hat{A}$ on $(\hat{A}, \hat{\Delta})$, respectively, given by the formulas:

- $\hat{\varepsilon}(\omega) = \omega(1)$,
- $\hat{S}(\omega)(a) = \omega(S(a))$,

where $\omega \in \hat{A}$ and $a \in A$. Hence $(\hat{A}, \hat{\Delta})$ is a Hopf $^*$-algebra.

Consider two finite-dimensional Hopf $^*$-algebras $(A_1, \Delta_1)$ and $(A_2, \Delta_2)$ and a Hopf $^*$-algebra isomorphism $F : A_1 \to A_2$. Then we can also dualize the mapping $F$ to the mapping $\hat{F} : \hat{A}_2 \to \hat{A}_1$ given by $\hat{F}(\omega) = F\omega$ for all $\omega \in \hat{A}_2$. It is an easy exercise to check that $\hat{F}$ is a Hopf $^*$-algebra isomorphism from $(\hat{A}_2, \hat{\Delta}_2)$ to $(\hat{A}_1, \hat{\Delta}_1)$.

Two Hopf $^*$-algebras $(A_1, \Delta_1)$ and $(A_2, \Delta_2)$ are called a *dual pair* if and only if there exists a non-degenerate bilinear form $\langle \cdot, \cdot \rangle : A_1 \times A_2 \to \mathbb{C}$ such that:

1. $\langle a_1 b_1, a_2 \rangle = \langle a_1 \otimes b_1, \Delta_2(a_2) \rangle$,
2. $\langle a_1, a_2 b_2 \rangle = \langle \Delta_1(a_1), a_2 \otimes b_2 \rangle$,
3. $\langle a_1^*, a_2 \rangle = \overline{\langle a_1, S_2(a_2)^* \rangle}$,
4. $\langle a_1, a_2^* \rangle = \overline{\langle S_1(a_1)^*, a_2 \rangle}$,
5. $\langle S_1(a_1), a_2 \rangle = \langle a_1, S_2(a_2) \rangle$,
6. $\langle a_1, 1 \rangle = \varepsilon_1(a_1)$ and $\langle 1, a_2 \rangle = \varepsilon_2(a_2)$

for all $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$. Equations 1 and 2 involve the extended bilinear form $\langle \cdot, \cdot \rangle : (A_1 \odot A_1) \times (A_2 \odot A_2) \to \mathbb{C}$ determined by the equality $\langle a_1 \odot b_1, a_2 \odot b_2 \rangle = \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle$ for all $a_1, b_1 \in A_1$, $a_2, b_2 \in A_2$. Uniqueness of the co-unit and the antipode implies that equations 3 to 6 are redundant. Invoking the non-degeneracy of $\langle \cdot, \cdot \rangle$, we see that $(A_1, \Delta_1)$ is commutative if and only if $(A_2, \Delta_2)$ is co-commutative.

The Hopf $^*$-algebras $(A, \Delta)$ and $(\hat{A}, \hat{\Delta})$ form a dual pair under the bilinear form $\langle \cdot, \cdot \rangle : A \times \hat{A} \to \mathbb{C}$ given by the formula $\langle a, \omega \rangle = \omega(a)$ for all $a \in A$, $\omega \in \hat{A}$. In fact, all dual pairs are of this form.

Suppose $(A, \Delta)$ is a finite quantum group with Haar state $h$. For $a \in A$ define $ah \in \hat{A}$ such that $(ah)(x) = h(xa)$, where $x \in A$. Since $h$ is faithful, the linear map $A \to \hat{A} : a \mapsto ah$ is injective and consequently bijective because $A$ and $\hat{A}$ have the same dimension. Thus $\hat{A} = \{\, ah \mid a \in A \,\}$.

We should point out that the above vector space isomorphism fails to be an isomorphism on the level of Hopf $^*$-algebras (it is not even multiplicative). However, we can use it to pull back the Hopf $^*$-algebra structure on the dual $(\hat{A}, \hat{\Delta})$ to $A$. In the case that $A = K(G)$ this pull-back Hopf $^*$-algebra is identical to the Hopf $^*$-algebra $(\mathbb{C}[G], \hat{\Delta})$.

Let $(\hat{A}, \hat{\Delta})$ be the dual Hopf $^*$-algebra of $(A, \Delta)$. Define the linear functional $\hat{h}$ on $\hat{A}$ by $\hat{h}(ah) = \varepsilon(a)$ for $a \in A$. Notice that $\hat{h}$ is well defined because $A \to \hat{A} : a \mapsto ah$ is a vector space isomorphism. We prove some properties for $\hat{h}$:

**Proposition 2.4** *Suppose $a \in A$ and $\xi \in \hat{A}$. Then we have $(\xi\hat{h})(ah) = \xi(S(a))$.*

**Proof:** Pick $b \in A$ such that $\xi = bh$. Using the $^*$-operation and the surjectivity of $T_2$, we see that there exist elements $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ in $A$ such that $a \otimes b = \sum_{i=1}^n \Delta(p_i)(q_i \otimes 1)$. Applying $m(S \otimes \iota)$ to this equality, we get $S(a)b = \sum_{i=1}^n \varepsilon(p_i)\, S(q_i)$. Inserting $x \in A$, we calculate

$$\begin{aligned}
((ah)\xi)(x) &= (ah \odot \xi)\Delta(x) \\
&= (h \odot h)(\Delta(x)(a \otimes b)) \\
&= \sum_{i=1}^n (h \odot h)(\Delta(xp_i)(q_i \otimes 1)) \\
&= \sum_{i=1}^n h(xp_i)\, h(q_i) \\
&= \sum_{i=1}^n h(xp_i)\, h(S(q_i)) \ .
\end{aligned}$$

Hence $(ah)\xi = \sum_{i=1}^n h(S(q_i))\; p_i h$, which implies that

$$\begin{aligned}
(\xi\hat{h})(ah) = \hat{h}((ah)\xi) &= \sum_{i=1}^n h(S(q_i))\, \hat{h}(p_i h) \\
&= \sum_{i=1}^n h(S(q_i))\, \varepsilon(p_i) = h(S(a)b) \\
&= \xi(S(a)) \ ,
\end{aligned}$$

as desired. □

As will be clear later, the following corollary is the quantum analogue of Plancherel's theorem.

**Corollary 2.5** *The formula $\hat{h}((ah)^*(ah)) = h(a^*a)$ holds for all $a \in A$.*

**Proof:** We have $(ah)^*(x) = \overline{(ah)(S(x)^*)} = \overline{h(S(x)^*a)} = h(a^*S(x)) = h\big(S(xS^{-1}(a^*))\big) = h(xS^{-1}(a^*))$ for all $x \in A$. It follows that $(ah)^* = S^{-1}(a^*)h$, which implies in turn that $\hat{h}((ah)^*(ah)) = \hat{h}((S^{-1}(a^*)h)(ah)) = ((ah)\hat{h})(S^{-1}(a^*)h) \overset{(*)}{=} (ah)\big(S(S^{-1}(a^*))\big) = h(a^*a)$, where we used the previous proposition in the equality labelled (*). □

**Proposition 2.6** *The map $\hat{h}$ is a Haar functional on $(\hat{A}, \hat{\Delta})$.*

**Proof:** Choose $b, c \in A$. As before, we can find elements $p_1, \ldots, p_n \in A$, $q_1, \ldots, q_n \in A$ such that $b \otimes c = \sum_{i=1} \Delta(p_i)(q_i \otimes 1)$. Applying $\varepsilon \odot \iota$ to this equation yields $\varepsilon(b)\, c = \sum_{i=1}^n p_i\, \varepsilon(q_i)$. So, for $x, y \in A$,

$$(\hat{\Delta}(bh)(1 \otimes ch))(x \otimes y) = (bh \otimes ch)((x \otimes 1)\Delta(y))$$
$$= (h \otimes h)((x \otimes 1)\Delta(y)(b \otimes c)) = \textstyle\sum_{i=1}^n (h \otimes h)((x \otimes 1)\Delta(yp_i)(q_i \otimes 1)).$$

Left invariance of $h$ now implies that $(\hat{\Delta}(bh)(1 \otimes ch))(x \otimes y) = \sum_{i=1}^n h(xq_i)\, h(yp_i)$ for every $x, y \in A$. Hence, $\hat{\Delta}(bh)(1 \otimes ch) = \sum_{i=1}^n q_i h \otimes p_i h$. Therefore we have $(\hat{h} \odot \iota)(\hat{\Delta}(bh)(1 \otimes ch)) = \sum_{i=1}^n \hat{h}(q_i h)\, p_i h = \sum_{i=1}^n \varepsilon(q_i)\, p_i h = \varepsilon(b)\, ch = \hat{h}(bh)\, ch$ and consequently $(\hat{h} \odot \iota)\hat{\Delta}(bh) = \hat{h}(bh)\, 1$, which shows the right invariance of $\hat{h}$.

Pick $a \in A$ such that $ah = \varepsilon$, so $h(a) = \varepsilon(1) = 1$. By the Cauchy-Schwarz inequality, $1 = |h(a)|^2 \leq h(1)\, h(a^*a)$, so $h(a^*a) \neq 0$. By Plancherel's formula, $\hat{h}(\varepsilon) = \hat{h}(\varepsilon^*\varepsilon) = \hat{h}((ah)^*(ah)) = h(a^*a) \neq 0$. □

Plancherel's formula shows that $\hat{h}$ is positive and faithful. Thus $\hat{A}$ is a Frobenius algebra and $(\hat{A}, \hat{\Delta})$ is a finite quantum group. Therefore we have duality within the category of finite quantum groups.

Consider the commutative and co-commutative Hopf $^*$-algebras $(K(G), \Delta)$ and $(\mathbb{C}[G], \hat{\Delta})$, respectively, associated to a finite group $G$. They constitute a dual pair under the bilinear form $\langle \cdot \mid \cdot \rangle : K(G) \times \mathbb{C}[G] \to \mathbb{C}$ given by

$$\langle f \mid g \rangle = \sum_{s \in G} f(s) g(s) = h(fg)$$

for all $f \in K(G)$ and $g \in \mathbb{C}[G]$, where $h$ is the Haar functional on $(K(G), \Delta)$.

Using this Haar functional $h$, we define a Hopf $^*$-algebra isomorphism $\theta : \mathbb{C}[G] \to \widehat{K(G)} : g \mapsto gh$. Then $\langle f \mid g \rangle = \langle f, \theta(g) \rangle$ for all $f \in K(G)$ and $g \in \mathbb{C}[G]$, where the latter form $\langle \cdot, \cdot \rangle$ is the one giving the duality between $(K(G), \Delta)$ and $(\widehat{K(G)}, \hat{\Delta})$ as described above.

Suppose that $G$ is an abelian finite group and denote by $\hat{G}$ the dual group of $G$. Recall that $\hat{G}$ is the set of group homomorphisms from $G$ to the circle $\mathbb{T}$, and that $\hat{G}$ is a group under the pointwise product.

Notice that $\hat{G}$ is a subset of $K(G)$. The Fourier transform $\mathcal{F} : \mathbb{C}[G] \to K(\hat{G})$ is given in terms of the duality $\langle \cdot \mid \cdot \rangle$ between $K(G)$ and $\mathbb{C}[G]$ by $\mathcal{F}(g)(\lambda) = \langle \lambda \mid g \rangle = h(\lambda g)$, where $g \in \mathbb{C}[G]$ and $\lambda \in \hat{G}$. The same formula extends $\mathcal{F}(g)$ from $\hat{G}$ to $K(G)$.

Notice that the Fourier transform $\mathcal{F} : \mathbb{C}[G] \to K(\hat{G})$ is an isomorphism of Hopf $^*$-algebras. We restrict ourselves to showing that $\mathcal{F}$ is a $^*$-homomorphism:

- Let $\lambda \in \hat{G}$. Because $\lambda$ is a group homomorphism, by definition of the co-multiplication $\Delta$ on $K(G)$, we get $\Delta(\lambda) = \lambda \otimes \lambda$. Now

$$\begin{aligned} \mathcal{F}(g_1 g_2)(\lambda) &= \langle \lambda \mid g_1 g_2 \rangle = \langle \Delta(\lambda) \mid g_1 \otimes g_2 \rangle = \langle \lambda \otimes \lambda \mid g_1 \otimes g_2 \rangle \\ &= \langle \lambda, g_1 \rangle \langle \lambda, g_2 \rangle = \mathcal{F}(g_1)(\lambda) \mathcal{F}(g_2)(\lambda) = (\mathcal{F}(g_1)\mathcal{F}(g_2))(\lambda) \end{aligned}$$

for all $g_1, g_2 \in \mathbb{C}[G]$ and $\lambda \in \hat{G}$.

• Observe that $S(\lambda) = \lambda^*$ for every $\lambda \in \hat{G}$. Thus

$$\mathcal{F}(g^*)(\lambda) = \langle \lambda \mid g^* \rangle = \overline{\langle S(\lambda)^* \mid g \rangle} = \overline{\langle \lambda \mid g \rangle} = \overline{\mathcal{F}(g)(\lambda)} = \mathcal{F}(g)^*(\lambda)$$

for all $g \in \mathbb{C}[G]$ and $\lambda \in \hat{G}$.

Recall that $K(G)$ and $\mathbb{C}[G]$ are equal as vector spaces (but not as algebras). Thus we may define $\tilde{\mathcal{F}} : K(G) \to K(\hat{G})$ by $\tilde{\mathcal{F}}(f) = \mathcal{F}(f)$, where $f \in K(G)$. The Plancherel Formula for a finite abelian group $G$ states that $\tilde{\mathcal{F}}$ is an isometry with respect to the $L^2$-norms given by the Haar measures on $G$ and $\hat{G}$ (if correctly scaled), i.e.

$$|G| \sum_{s \in G} |f(s)|^2 = \sum_{\lambda \in \hat{G}} |\tilde{\mathcal{F}}(f)(\lambda)|^2$$

for all $f \in K(G)$. We prove that the formula in Corollary 2.5 takes this form whenever $(A, \Delta) = (K(G), \Delta)$, where now $G$ is a finite abelian group:

Denote by $h'$ the Haar functional on $(K(\hat{G}), \Delta)$. Since $\theta\mathcal{F}^{-1} : K(\hat{G}) \to \widehat{K(G)}$ is a Hopf $^*$-algebra isomorphism, it follows by uniqueness of the Haar functional $h'$ that $\hat{h}\theta\mathcal{F}^{-1} = ch'$ for some $c \in \mathbb{C}$. Now observe that

$$\sum_{s \in G} |f(s)|^2 = h(f^* f) = \hat{h}((fh)^*(fh)) = \hat{h}\theta\mathcal{F}^{-1}(\tilde{\mathcal{F}}(f)^* \tilde{\mathcal{F}}(f))$$

$$= ch'(\tilde{\mathcal{F}}(f)^* \tilde{\mathcal{F}}(f)) = c \sum_{\lambda \in \hat{G}} |\tilde{\mathcal{F}}(f)(\lambda)|^2$$

for all $f \in K(G)$. It remains to prove that $c = |G|^{-1}$. To this end insert $f = 1$ in the formula above and note that $\tilde{\mathcal{F}}(1)(\lambda) = h(\lambda)$, so

$$|G| = \sum_{s \in G} |1(s)|^2 = c \sum_{\lambda \in \hat{G}} |h(\lambda)|^2.$$

Now

$$h(\lambda)1 = (h \odot \iota)\Delta(\lambda) = h(\lambda)\lambda$$

for all $\lambda \in \hat{G}$, so $h(\lambda) = 0$ for all $\lambda \neq 0$. Thus $|G| = c\, h(1)^2 = c\,|G|^2$ as desired.

Given a finite-dimensional Hopf $^*$-algebra $(A, \Delta)$, we may, of course, form the dual of the dual Hopf $^*$-algebra $(\hat{A}, \hat{\Delta})$ to get the double dual Hopf $^*$-algebra $(\hat{\hat{A}}, \hat{\hat{\Delta}})$. In fact, since $A$ is finite-dimensional, we have a canonical isomorphism $\pi : A \to \hat{\hat{A}}$ of vector spaces given by $\pi(a)(\omega) = \omega(a)$ for all $a \in A$ and $\omega \in \hat{A}$. A closer investigation shows that $\pi$ is a Hopf $^*$-algebra isomorphism. It follows that if $(A, \Delta)$ is a finite-dimensional Hopf $^*$-algebra, then $A$ is a Frobenius algebra if and only if $\hat{A}$ is a Frobenius algebra. Also the formula in Proposition 2.4 can be rewritten in terms of the dual forms $\langle \cdot, \cdot \rangle_A : A \times \hat{A} \to \mathbb{C}$ and $\langle \cdot, \cdot \rangle_{\hat{A}} : \hat{A} \times \hat{\hat{A}} \to \mathbb{C}$, namely,

$$\langle \pi^{-1}(\omega \hat{h}), ah \rangle_A = \langle S(a), \omega \rangle_A$$

and

$$\langle ah, \omega \hat{h} \rangle_{\hat{A}} = \langle \hat{S}(\omega), \pi(a) \rangle_{\hat{A}},$$

for all $a \in A$ and $\omega \in \hat{A}$.

Consider again a finite abelian group $G$. Taking into account the canonical Hopf $^*$-algebra isomorphisms introduced in the discussion above, we arrive at the following diagram:

$$\begin{tikzcd}
K\left(\hat{\hat{G}}\right) \arrow[r, "\mathcal{F}^{-1}"] \arrow[d, "\hat{\mathcal{P}}"'] & \mathbb{C}[\hat{G}] \arrow[r, "\theta"] & \widehat{K(\hat{G})} \arrow[d, "\hat{\mathcal{F}}"] \\
K(G) & \widehat{\widehat{K(G)}} \arrow[l, "\pi^{-1}"'] & \widehat{\mathbb{C}[G]} \arrow[l, "\hat{\theta}^{-1}"']
\end{tikzcd}$$

where we have defined the isomorphism $\hat{\mathcal{P}} : K\left(\hat{\hat{G}}\right) \to K(G)$ in such a way that the diagram commutes. The Hopf $^*$-algebra isomorphism $\hat{\mathcal{P}}$ has to be the transpose of a group isomorphism $\mathcal{P} : G \to \hat{\hat{G}}$, i.e. $\hat{\mathcal{P}}(f) = f\mathcal{P}$ for all $f \in K(\hat{\hat{G}})$. This is Pontryagin's duality theorem for finite abelian groups. Of course,

$\hat{\hat{G}} \simeq \hat{G} \simeq G$ via the classification of finite abelian groups, but these isomorphisms are not canonical.

A tedious but straightforward calculation shows that $\mathcal{P}$ is given by the well known formula appearing in Pontryagin's duality theorem for finite abelian groups, i.e. $\mathcal{P}(s)(\omega) = \omega(s)$ for all $s \in G$ and $\omega \in \hat{G}$.

**Remark 2.7** We state some general facts about finite quantum groups. It can be shown, [37], that for a finite quantum group $(A, \Delta)$ the Haar functional $h$ is tracial, that is, $h(ab) = h(ba)$ for all $a, b \in A$. Furthermore, the antipode is involutive and $^*$-preserving, so a finite quantum group is a Kac algebra (see [8] and Section 3, and also Section 7 (part II)). Any finite-dimensional Frobenius- or C$^*$-algebra is a direct sum of full matrix algebras, and the irreducible $^*$-representations of such an algebra are all obtained by projecting down on any of the factors in the product. The co-unit of a finite quantum group is a 1-dimensional $^*$-representation of $A$, and therefore $A$ must contain a copy of $\mathbb{C}$.

The smallest non-abelian group is the group of permutations of three elements, which consists of 6 elements. The corresponding finite quantum group is a 6-dimensional commutative and non-cocommutative Hopf $^*$-algebra. The following (see exercise 7 on p. 68 in [14]) is an example of a 4-dimensional Hopf algebra which is neither commutative nor co-commutative:

Denote by $A$ the universal unital algebra generated by two elements $t$ and $x$ satisfying the relations

$$t^2 = 1 \qquad x^2 = 0 \qquad xt = -tx \ .$$

The following formulas:

$$\Delta(t) = t \otimes t \qquad \Delta(x) = 1 \otimes x + x \otimes t$$

and

$$\begin{array}{rclcrcl} \varepsilon(t) & = & 1 & \qquad & S(t) & = & 0 \\ \varepsilon(x) & = & 0 & \qquad & S(x) & = & tx \ . \end{array}$$

define a co-multiplication $\Delta$, a co-unit $\varepsilon$ and an antipode $S$ on $A$.

However, this Hopf algebra is not a quantum group because there is no $^*$-operation turning it into a Frobenius algebra. In fact, it is easy to see that the antipode $S$ satisfies $S^4 = \iota$ and that it is not involutive. The lowest dimensional finite quantum group which is neither commutative nor co-commutative is the 'historic' example due to G. I. Kac and V. G. Paljutkin, published in Russian in 1965. The algebra $A$ for this quantum group is 8-dimensional and is $^*$-isomorphic to $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$. It admits only one non-trivial Hopf $^*$-algebra structure, [9].

## 3. Compact Quantum Groups

### 3.1. Definition and Examples of Compact Quantum Groups

The definition we adopt of a compact quantum group is by now widely accepted. It is due to Woronowicz [35] , who also defined its immediate predecessor: a compact matrix pseudo-group [37]. A compact matrix pseudo-group is a generalization of a compact matrix group. Its definition distinguishes a fundamental co-representation, a role played by the identity representation in the matrix group case. It is well known that every compact Lie group has an injective finite-dimensional representation. Therefore, compact matrix pseudogroups can be regarded as quantum analogues of compact Lie groups.

Although the co-multiplication is defined on the C$^*$-algebra level, this is not the case for the co-unit and antipode. They may not be bounded. However, we shall see that there always exists a Hopf $^*$-algebra which is dense in the C$^*$-algebra. In the classical case this is the algebra of regular functions on the group. For matrix groups it is the $^*$-algebra generated by the co-ordinate functions, which by the Stone-Weierstrass Theorem is dense in the C$^*$-algebra of continuous functions on the group. One way to overcome the difficulties with a co-unit and a co-inverse not everywhere defined, is to invoke the maps $T_1$ and $T_2$ discussed in the previous section.

A note on terminology: when considering the tensor product $A \otimes B$ of C*-algebras $A$ and $B$, we shall always take it to be the C*-algebraic completion of the algebraic tensor product $A \odot B$ with respect to the minimal tensor product norm.

**Definition 3.1.1** *A compact quantum group is a pair* $(A, \Delta)$, *where* $A$ *is a unital C*-algebra and* $\Delta : A \to A \otimes A$ *is a unital* $^*$*-homomorphism such that:*

1. $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$,
2. $\Delta(A)(A \otimes 1)$ *is a dense subspace of* $A \otimes A$,
3. $\Delta(A)(1 \otimes A)$ *is a dense subspace of* $A \otimes A$.

In this definition, $\Delta(A)(A \otimes 1)$ is the linear span of the set $\{\, \Delta(a)(b \otimes 1) \mid a, b \in A \,\}$ (and similarly for $\Delta(A)(1 \otimes A)$).

Clearly, all finite quantum groups are compact quantum groups because, as we have seen, the maps $T_1$ and $T_2$ are bijective. Now we introduce the motivating example generalizing the finite quantum group $(K(G), \Delta)$, where $G$ is a finite group.

**Example 3.1.2** Suppose $G$ is a compact topological group and let $C(G)$ denote the set of continuous functions on $G$. It is a unital C*-algebra under the obvious pointwise-defined algebraic operations and the uniform norm (cf. the $^*$-algebra $K(G)$ when $G$ is finite). The linear map $\pi : C(G) \odot C(G) \to C(G \times G)$ determined by $\pi(f \otimes g)(s,t) = f(s)\,g(t)$ for $f, g \in C(G)$ and $s, t \in G$ has a unique continuous extension to a $^*$-isomorphism from $C(G) \otimes C(G)$ to $C(G \times G)$. We shall use this $^*$-isomorphism to identify $C(G) \otimes C(G)$ with $C(G \times G)$.

Define the unital $^*$-homomorphism $\Delta$ from $C(G)$ into $C(G) \otimes C(G)$ by setting $\Delta(f)(s,t) = f(st)$ for all $f \in C(G)$ and $s, t \in G$. The associativity of the group multiplication is then reflected in the co-associativity of $\Delta$, i.e. $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.

Define the $^*$-homomorphism $T_1$ from $C(G) \otimes C(G)$ to $C(G) \otimes C(G)$ such that $T_1(f)(s,t) = f(st, t)$ for $f \in C(G) \otimes C(G)$ and $s, t \in G$. As in the finite group case, this map is invertible with inverse given by $T_1^{-1}(f)(s,t) = f(st^{-1}, t)$ for $f \in C(G) \otimes C(G)$ and $s, t \in G$. Thus by continuity, $T_1(C(G) \odot C(G))$ is dense in

$C(G) \otimes C(G)$. Since $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$ for all $a, b \in C(G)$, we have proved axiom 3 in Definition 3.1.1. Axiom 2 is proved similarly. Hence, $(C(G), \Delta)$ is a compact quantum group.

The two $^*$-homomorphisms $S : C(G) \to C(G)$ and $\varepsilon : C(G) \to \mathbb{C}$ defined by the formulas $S(f)(s) = f(s^{-1})$ and $\varepsilon(f) = f(e)$, where $f \in C(G)$ and $s \in G$, play the role of the antipode and the co-unit on $(C(G), \Delta)$, respectively. They are obviously everywhere defined and bounded and furthermore, $S$ is an involutive $^*$-isomorphism. Nothing of this is true in the general case and this poses a challenging problem.

The pair $(C(G), \Delta)$ thus resembles very much the notion of a Hopf $^*$-algebra in the sense of Definition 2.1, in that we just have to replace the algebraic tensor product $\odot$ with the topological tensor product $\otimes$.

The previous example exhausts all compact quantum groups for which the C$^*$-algebra is commutative. The proof, which essentially uses Gelfand's representation theorem for commutative C*-algebras, boils down to showing that a compact semi-group satisfying the cancellation laws is a compact group, a well known fact. The next example exhausts all co-commutative compact quantum groups for which the Haar state is faithful. We give a proof of this fact at the end of this section.

**Example 3.1.3** We now consider $(\mathbb{C}[G], \hat{\Delta})$, where $G$ is a discrete group. In Section 2 we showed that $(\mathbb{C}[G], \hat{\Delta})$ is a unital Hopf $^*$-algebra with a positive and faithful Haar functional $\hat{h}$.

In order to get a compact quantum group, we need to show that the unital $^*$-algebra $\mathbb{C}[G]$ admits a C$^*$-norm. We use the Haar functional to introduce a Hilbert space and represent the unital $^*$-algebra $\mathbb{C}[G]$ injectively as bounded operators on this Hilbert space.

The inner product $(\cdot \mid \cdot)$ on $\mathbb{C}[G]$ is defined by the equations $(f \mid g) = \hat{h}(g^* f)$ for $f, g \in \mathbb{C}[G]$. Denote the corresponding Hilbert space completion of $\mathbb{C}[G]$ by $L^2(\hat{h})$ and the norm on it by $\|\cdot\|_2$. Notice that $\|f\|_2^2 = \sum_{s \in G} |f(s)|^2$ for $f \in \mathbb{C}[G]$.

Functions $f$ in $\mathbb{C}[G]$ can be represented as linear operators $L_f$ in $L^2(\hat{h})$ given by left multiplication on $\mathbb{C}[G]$: $L_f(g) = fg$ for all $g \in \mathbb{C}[G]$. We proceed by showing that the operators $L_f$, where $f \in \mathbb{C}[G]$, are bounded on the dense subspace $\mathbb{C}[G]$ of $L^2(\hat{h})$, and thus have unique extensions to bounded operators $\pi(f)$ on $L^2(\hat{h})$: To this end, recall that $\mathbb{C}[G]$ comes with a linear basis $(\delta_s)_{s\in G}$, where $\delta_s^*\delta_s = \delta_e$ for $s \in G$. Hence $\|L_{\delta_s}(g)\|_2^2 = \|\delta_s g\|_2^2 = \hat{h}((\delta_s g)^*(\delta_s g)) = \hat{h}(g^*\delta_s^*\delta_s g) = \hat{h}(g^*\delta_e g) = \hat{h}(g^*g) = \|g\|_2^2$ for all $r, s \in G$. We have shown that $L_{\delta_s}$ is isometric, hence bounded of norm one. The result that $L_f$ is bounded for an arbitrary $f \in \mathbb{C}[G]$ now follows by the triangle inequality and the fact that the elements $(\delta_s)_{s\in G}$ constitute a linear basis for $\mathbb{C}[G]$.

It is not difficult to check that the map $\mathbb{C}[G] \to B(L^2(\hat{h})) : f \mapsto \pi(f)$ is a unital injective $^*$-homomorphism. It is essentially the GNS-representation of the functional $\hat{h}$. The operator norm $\|\cdot\|$ on $B(L^2(\hat{h}))$ (the bounded operators on $L^2(\hat{h})$) is a C$^*$-norm, so we get a C$^*$-norm $\|\cdot\|_r$ on $\mathbb{C}[G]$ by defining $\|f\|_r = \|\pi(f)\|$, for $f \in \mathbb{C}[G]$. Let $C_r^*(G)$ denote the C$^*$-algebra completion of $\mathbb{C}[G]$ with respect to this norm. By definition of $C_r^*(G)$, there exists a unique unital faithful representation $\pi : C_r^*(G) \to B(L^2(\hat{h}))$ which extends the map $\mathbb{C}[G] \to B(L^2(\hat{h})) : f \mapsto \pi(f)$.

We prove that the co-multiplication $\hat{\Delta}$ on $\mathbb{C}[G]$ is bounded with respect to $\|\cdot\|_r$, and therefore we have an extension to a co-multiplication $\hat{\Delta}_r$ on $C_r^*(G)$:

Let $f \in \mathbb{C}[G]$. Consider a complex-valued function $F$ on $G\times G$, with finite support, regarded as an element of $L^2(\hat{h})\otimes L^2(\hat{h})$. We show that

$$\|(\pi \odot \pi)(\hat{\Delta}(f))\, F\|_2 \leq \|\pi(f)\|\, \|F\|_2,$$

which is sufficient to conclude that $\hat{\Delta}(f)$ has $\|\cdot\|_r$-norm less than $\|f\|_r$. It is not difficult to see that $\big((\pi \odot \pi)(\hat{\Delta}(f))\, F\big)(p, q) = \sum_{s\in G} f(s)\, F(s^{-1}p, s^{-1}q)$ for $p, q \in G$ (it is enough to check this formula for $f$ of the form $\delta_t$, $t \in G$). Thus

$$\begin{aligned}
\|(\pi \odot \pi)(\hat{\Delta}(f))\, F\|_2^2 &= \sum_{p,q \in G} |\sum_{s \in G} f(s)\, F(s^{-1}p, s^{-1}q)\,|^2 \\
&= \sum_{q \in G} \sum_{p \in G} |\sum_{s \in G} f(s)\, F(s^{-1}p, s^{-1}pq)\,|^2 \\
&= \sum_{q \in G} (\sum_{p \in G} |\,(f\, F(\cdot, \cdot\, q))(p)\,|^2\,) \\
&= \sum_{q \in G} \|\pi(f)\, F(\cdot, \cdot\, q)\|_2^2 \leq \sum_{q \in G} \|\pi(f)\|^2\, \|F(\cdot, \cdot\, q)\|_2^2 \\
&= \|\pi(f)\|^2 \sum_{p,q \in G} |F(p, pq)|^2 \\
&= \|\pi(f)\|^2 \sum_{p,q \in G} |F(p, q)|^2 = \|\pi(f)\|^2\, \|F\|_2^2\,.
\end{aligned}$$

That $(C_r^*(G), \hat{\Delta}_r)$ is a compact quantum group now follows, since the reduced C$^*$-algebra $C_r^*(G)$ contains the dense Hopf $^*$-algebra $(\mathbb{C}[G], \hat{\Delta})$.

A crucial result in the theory of compact quantum groups is the existence of the Haar state. This fact was proven by Woronowicz in the separable case (see [31]). The general case was proven by A. Van Daele (see [30]). A proof under weaker conditions can be found in [20]. Precisely formulated, the existence result says that:

**Theorem 3.1.4** *Consider a compact quantum group* $(A, \Delta)$. *There exists a unique state* $h$ *on* $A$ *such that* $(h \otimes \iota)\Delta(a) = (\iota \otimes h)\Delta(a) = h(a)\, 1$ *for all* $a \in A$. *The functional* $h$ *is called the Haar state on* $(A, \Delta)$.

The proof of the uniqueness of the Haar state is essentially trivial. The following result holds: Let $h_1, h_2$ be two states on $A$ such that $(h_1 \otimes \iota)\Delta(a) = h_1(a)\, 1$ and $(\iota \otimes h_2)\Delta(a) = h_2(a)\, 1$ for $a \in A$. Then $h_1 = h_2$.

Unlike the classical case, the Haar state does not have to be faithful. We give an example where faithfulness does not hold.

**Example 3.1.5** Let $G$ be a discrete group and consider again the unital Hopf $^*$-algebra $(\mathbb{C}[G], \hat{\Delta})$. We endow $\mathbb{C}[G]$ with a universal C$^*$-norm $\|\cdot\|_u$ by defining $\|x\|_u$ to be the supremum of the set

$$\{\|\theta(x)\| \mid K \text{ a Hilbert space, } \theta \text{ a unital } {}^*\text{-representation of } \mathbb{C}[G] \text{ on } K\}$$

for each $x \in \mathbb{C}[G]$. Consider $\delta_s$ for $s \in G$ and let $\theta$ be a unital $^*$-representation of $\mathbb{C}[G]$ on a Hilbert space $K$. Then

$$\|\theta(\delta_s)\|^2 = \|\theta(\delta_s)^* \, \theta(\delta_s)\| = \|\theta(\delta_s^* \delta_s)\| = \|\theta(\delta_e)\| = \|1\| = 1,$$

so $\|\delta_s\|_u \leq 1$. Finiteness of $\|x\|_u$ for arbitrary $x \in \mathbb{C}(G)$ now follows since $(\delta_s)_{s \in G}$ is a linear basis for $\mathbb{C}[G]$.

Remembering that, in the definition of $\|x\|_u$, the set we are taking the supremum over contains the number $\|x\|_r$ from Example 3.1.2, we deduce that $\|\cdot\|_u$ is a C$^*$-norm on $\mathbb{C}[G]$ (not merely a C*-semi-norm). Denote the C$^*$-algebra completion of $\mathbb{C}[G]$ with respect to $\|\cdot\|_u$ by $C^*(G)$. This unital C$^*$-algebra is obviously universal in the following sense: Any unital $^*$-representation $\theta$ of $C[G]$ on a Hilbert space has a unique extension to a (bounded) $^*$-representation of $C^*(G)$.

Let $U : G \to B(K)$ be a strongly continuous unitary representation of the group $G$ and define the map $\theta_U : \mathbb{C}[G] \to B(K)$ by $\theta_U(f) = \sum_{s \in G} f(s)U(s)$ for $f \in \mathbb{C}[G]$. It is easily seen that $\theta_U$ is a unital $^*$-representation and that $\theta_U(\delta_s) = U(s)$ for $s \in G$. Keeping this in mind, we get a 1-1 correspondence between strongly continuous unitary representations of the group $G$ and unital $^*$-representations of the C$^*$-algebra $C^*(G)$.

It is well known that the unital C$^*$-algebra $C^*(G) \otimes C^*(G)$ can be faithfully represented on a Hilbert space $K$ — this is true for any C$^*$-algebra. Considering the following embeddings $\mathbb{C}[G] \odot \mathbb{C}[G] \subseteq C^*(G) \otimes C^*(G) \hookrightarrow B(K)$, we may view $\Delta$ as a unital $^*$-representation of $\mathbb{C}[G]$ on $K$. Thus it has a bounded extension $\hat{\Delta}_u : C^*(G) \to B(K)$. Combining the facts that $\mathbb{C}[G]$ is dense in $C^*(G)$, that $\hat{\Delta}_u$ is continuous and that $C^*(G) \otimes C^*(G)$ is closed in $B(K)$, we see that $\hat{\Delta}_u(C^*(G)) \subseteq C^*(G) \otimes C^*(G)$. So we get a unital $^*$-homomorphism $\hat{\Delta}_u : C^*(G) \to C^*(G) \otimes C^*(G)$. Hence, the pair $(C^*(G), \hat{\Delta}_u)$ is a compact quantum group.

For similar reasons, there exists a unital surjective $^*$-homomorphism $\pi_u : C^*(G) \to C_r^*(G)$ such that $\pi_u(f) = f$ for all $f \in \mathbb{C}[G]$. This $^*$-homomorphism is an isomorphism if and only if $G$ is amenable (which is true if $G$ is abelian or finite).

Let $\hat{h}_r$ be the vector state on $C_r^*(G)$ given by $\hat{h}_r(x) = (\pi(x)\delta_e \mid \delta_e)$ for $x \in C^*(G)$, where $\pi$ is the representation of $C_r^*(G)$ on $B(L^2(\hat{h}))$ introduced in Example 3.1.3. Its restriction to $\mathbb{C}[G]$ is of course the Haar functional $\hat{h}$ on $\mathbb{C}[G]$, so by continuity, $\hat{h}_r$ is the Haar state on $(C_r^*(G), \hat{\Delta}_r)$. By definition of $\hat{h}$, we see that $\hat{h}(\delta_s\delta_t) = \hat{h}(\delta_{st}) = \hat{h}(\delta_{ts}) = \hat{h}(\delta_t\delta_s)$ for $s, t \in G$ so by linearity and continuity, $\hat{h}_r$ is tracial, i.e. $\hat{h}_r(xy) = \hat{h}_r(yx)$ for all $x, y \in C_r^*(G)$. Since $\pi(\delta_s)\delta_e = \delta_s$ for all $s \in G$, we also see that $\pi(C_r^*(G))\,\delta_e$ is dense in $H$.

Now take any $x \in C_r^*(G)$ such that $\hat{h}_r(x^*x) = 0$. Then we get for all $y \in C_r^*(G)$, that

$$\begin{aligned}(\pi(x)(\pi(y)\delta_e) \mid \pi(x)(\pi(y)\delta_e)) &= \hat{h}_r(y^*x^*xy) = \hat{h}_r(x^*xyy^*) \\ &= \hat{h}_r\big(\,(x^*x)^{\frac{1}{2}}\,yy^*(x^*x)^{\frac{1}{2}}\,\big) \leq \|y\|^2\,\hat{h}_r(x^*x) = 0\ ,\end{aligned}$$

which implies that $\pi(x)(\pi(y)\delta_e) = 0$, so $\pi(x) = 0$ and therefore $x = 0$. We have shown that the Haar state $\hat{h}_r$ on $(C_r^*(G), \hat{\Delta}_r)$ is faithful.

Since $(\pi_u \otimes \pi_u)\hat{\Delta}_u = \hat{\Delta}_r\pi_u$, we conclude by uniqueness that the Haar state $\hat{h}_u$ on $(C^*(G), \hat{\Delta}_u)$ is given by $\hat{h}_u = \hat{h}_r\,\pi_u$. It is now clear that $\hat{h}_u$ is faithful if and only if the group $G$ is amenable (the classical example of a non-amenable discrete group is the free group on two generators).

As for the co-multiplication, one may argue that there exists a unique bounded unital $^*$-homomorphism $\hat{\varepsilon}_u : C^*(G) \to \mathbb{C}$ which extends $\hat{\varepsilon}$, the co-unit of $(\mathbb{C}[G], \hat{\Delta})$, so $\hat{\varepsilon}$ is also bounded with respect to $\|\cdot\|_r$ when $G$ is amenable.

We now give the standard example of a compact quantum group that is neither commutative nor co-commutative (more precisely, it is a one-parameter family of compact quantum groups).

Thus it is not included in the cases $(C(G), \Delta)$ for $G$ a compact group nor $(\mathbb{C}[G], \hat{\Delta})$ for $G$ a discrete group described above. In fact, it was also one of the first examples not included in the category of Kac algebras. The example is due to Woronowicz (see [38]) and is a 'deformation' of the special unitary group SU(2). Woronowicz called it twisted SU(2) (due to a twist in the determinant) and we denote it by $\mathrm{SU}_q(2)$, where $q$ is a deformation parameter. This example was discovered among a huge range of examples found by Drinfeld (see [7]) and M. Jimbo (see [12]), namely as a deformation $\mathcal{U}_q(\mathrm{su}(2))$ of the universal enveloping algebra $\mathcal{U}(\mathrm{su}(2))$ of the Lie algebra su(2) associated to the Lie group SU(2). The connection between these two dual approaches was first recognized and established by Y. S. Soibelman and L. L. Vaksman (see [28]) and M. Rosso (see [24]). They used representations of $\mathcal{U}_q(\mathrm{su}(2))$ as the linking mechanism (see Section 6 (part II)).

Soibelman & Vaksman also developed the harmonic analysis on twisted SU(2). They showed [28] how little Jacobi $q$-polynomials could be given a geometric interpretation on a 'quantum space'. Later (see [17]) S. Levendorski & Soibelman generalized the method of representations to generate examples of compact matrix pseudogroups from Drinfeld's and Jimbo's deformations of the simple Lie algebras.

This example is also typical of how 'proper' quantum groups (i.e. those that are neither commutative nor co-commutative) are constructed. In most cases, these examples tend to appear rather ad hoc, although deformation theory gives certain restrictions on the possibilities.

**Example 3.1.6** Let $q \in [-1, 1] \backslash \{0\}$. Define $\mathcal{A}$ to be the universal unital $^*$-algebra generated by two elements $\alpha$ and $\gamma$ satisfying the relations:

$$\alpha^*\alpha + \gamma^*\gamma = 1 \qquad\qquad \alpha\,\alpha^* + q^2\,\gamma\,\gamma^* = 1$$

$$\gamma\,\gamma^* = \gamma^*\gamma \qquad q\,\gamma\,\alpha = \alpha\,\gamma \qquad q\,\gamma^*\alpha = \alpha\,\gamma^* \ .$$

The universality of $\mathcal{A}$ assures us that there exist a unital $^*$-homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \odot \mathcal{A}$, a unital $^*$-homomorphism

$\varepsilon : \mathcal{A} \to \mathbb{C}$ and an unital anti-homomorphism $S : \mathcal{A} \to \mathcal{A}$ such that:

$$\Delta(\alpha) = \alpha \otimes \alpha - q\,\gamma^* \otimes \gamma \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$$

$$S(\alpha) = \alpha^* \qquad S(\alpha^*) = \alpha \qquad S(\gamma) = -q\,\gamma \qquad S(\gamma^*) = -q^{-1}\,\gamma^*$$

$$\varepsilon(\alpha) = 1 \qquad \varepsilon(\gamma) = 0\ .$$

The co-associativity of $\Delta$ and condition 2.1.1 of Definition 2.1 follow by inspection on the generators $\alpha$ and $\gamma$ (remember that $\Delta$ and $\varepsilon$ are $^*$-homomorphisms). Since the multiplication $m$ is not multiplicative and $S$ is anti-multiplicative, we have to handle condition 2.1.2 of Definition 2.1 differently. Define the linear subspace

$$\mathcal{A}_0 = \{\, a \in \mathcal{A} \mid m(S \odot \iota)\Delta(a) = \varepsilon(a)\,1 = m(\iota \odot S)\Delta(a)\,\}\ .$$

It is not difficult to check that $\mathcal{A}_0$ is a unital subalgebra of $\mathcal{A}$ which contains the elements $\alpha, \alpha^*, \gamma, \gamma^*$. Since these elements generate $\mathcal{A}$ as a unital algebra, we see that $\mathcal{A} = \mathcal{A}_0$. Hence $(\mathcal{A}, \Delta)$ is a Hopf $^*$-algebra with co-unit $\varepsilon$ and antipode $S$.

Notice that when $q = 1$, the algebra $\mathcal{A}$ is commutative (but not co-commutative). Recall that $\mathrm{SU}(2)$ is defined to be the group

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix} \mid a, c \in \mathbb{C} \text{ such that } |a|^2 + |c|^2 = 1 \right\}\ .$$

Introduce the coordinate functions $\alpha', \gamma' \in C(\mathrm{SU}(2))$ given by

$$\alpha' \begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix} = a \qquad \text{and} \qquad \gamma' \begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix} = c$$

for all $a, c \in \mathbb{C}$ such that $|a|^2 + |c|^2 = 1$. Denote by $\mathrm{Pol}(\mathrm{SU}(2))$ the unital $^*$-subalgebra of $C(\mathrm{SU}(2))$ generated by $\alpha'$ and $\gamma'$. By the Stone-Weierstrass Theorem, the $^*$-algebra $\mathrm{Pol}(\mathrm{SU}(2))$ is dense in the C$^*$-algebra $C(\mathrm{SU}(2))$. Furthermore observe that the coordinate functions $\alpha'$ and $\gamma'$ satisfy the same relations as $\alpha$ and $\gamma$ do (when $q = 1$). Therefore we have a surjective unital

$^*$-homomorphism $\theta : \mathcal{A} \to \mathrm{Pol}(\mathrm{SU}(2))$ such that $\theta(\alpha) = \alpha'$ and $\theta(\gamma) = \gamma'$. The identity

$$\theta(x) \begin{pmatrix} \lambda(\alpha) & \lambda(-\gamma^*) \\ \lambda(\gamma) & \lambda(\alpha^*) \end{pmatrix} = \lambda(x) \ ,$$

where $\lambda$ is a $^*$-character on $\mathcal{A}$, is easily checked on the generators and thus holds for all $x \in \mathcal{A}$. Hence $\theta(x) = 0 \Leftrightarrow \lambda(x) = 0$ for all $^*$-characters $\lambda$ on $\mathcal{A}$. So $\theta$ is injective if and only if the $^*$-characters separate elements in $\mathcal{A}$. It is well known that the $^*$-characters separate elements in a commutative C$^*$-algebra and by restriction, they supply a separating family of $^*$-characters for $^*$-subalgebras. In [38], Woronowicz proved that $\mathcal{A}$ has a universal C$^*$-algebra envelope $A$ by constructing enough unital $^*$-representations and working with a Hamel basis in $\mathcal{A}$. Hence we may identify $\mathcal{A}$ with $\mathrm{Pol}(\mathrm{SU}(2))$. In this way, we have recovered the compact group $\mathrm{SU}(2)$, and the Hopf $^*$-algebra of coordinate functions $\mathrm{Pol}(\mathrm{SU}(2))$ is isomorphic to the universal algebra $\mathcal{A}$.

Let us now proceed to the more interesting case when $q \neq 1$. In this case, $(\mathcal{A}, \Delta)$ is neither commutative nor co-commutative. In order to obtain a quantum group, we need a C$^*$-norm on $\mathcal{A}$ which makes $\Delta : \mathcal{A} \to \mathcal{A} \odot \mathcal{A}$ continuous.

Mimicking the construction for the norm on $C^*(G)$ in Example 3.1.5, we introduce a universal norm $\| \cdot \|_u$ on $\mathcal{A}$ by setting $\|a\|_u$ to be the supremum of the set

$$\{\, \|\theta(a)\| \mid K \text{ a Hilbert space, } \theta \text{ a unital } ^*\text{-representation of } \mathcal{A} \text{ on } K\}$$

for all $a \in \mathcal{A}$. The boundedness of the co-multiplication with respect to this norm is then immediate. However, it requires some argument to see that this really defines a norm.

The boundedness of $\|a\|_u$ for any element $a \in \mathcal{A}$ is proved by verifying it on the generators $\alpha$ and $\gamma$ (this reduction is justified by the triangle inequality, the submultiplicativity and the $^*$-invariance of $\| \cdot \|_u$). But the relation $\alpha^*\alpha + \gamma^*\gamma = 1$ gives $\|\alpha\|_u \leq 1$ and $\|\gamma\|_u \leq 1$.

As usual, the property $\|x\|_u = 0 \Leftrightarrow x = 0$ is the most difficult to prove. To prove it, it suffices to produce an injective unital $^*$-representation of $\mathcal{A}$: let $H$ be a separable Hilbert space with orthonormal basis $(\, e_{km} \mid k \in \mathbb{N} \cup \{0\}, m \in \mathbb{Z}\,)$ and define the $^*$-representation $\pi : \mathcal{A} \to B(H)$ by the formulas:

$$\pi(\alpha)e_{km} = \sqrt{1-q^{2k}}\, e_{k-1,m} \qquad \text{and} \qquad \pi(\gamma)e_{km} = q^k\, e_{k,m+1}\ ,$$

where we put $e_{-1,m} = 0$. The $^*$-representation is well defined because the operators $\pi(\alpha)$,$\pi(\gamma)$ satisfy the same relations as $\alpha$,$\gamma$, respectively, so we can appeal to the universal property of $\mathcal{A}$. For a proof of the injectivity of $\pi$, see [21].

The C$^*$-algebra completion $A_u$ of $\mathcal{A}$ with respect to $\|\cdot\|_u$ together with the continuous extension $\Delta_u : A_u \to A_u \otimes A_u$ of $\Delta$ form a pair $(A_u, \Delta_u)$ that constitutes a compact quantum group.

The representation $\pi$ allows one to express the Haar state $h$ on $(A_u, \Delta_u)$ by the formula

$$h(a) = (1-q^2) \sum_{k=0}^{\infty} q^{2k}\, \langle \pi_u(a) e_{k,0}, e_{k,0} \rangle$$

for all $a \in A$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product on $H$ and $\pi_u : A_u \to B(H)$ denotes the continuous extension of $\pi$. It is generally true that Haar functionals on Hopf $^*$-algebras are faithful (see [1]), so the restriction of $h$ to $\mathcal{A}$ is faithful. It is possible, but highly non-trivial, to show that $h$ is faithful on the C$^*$-algebra $A_u$ (see [21]). However, we stress that the Haar state is not tracial in this case, and for this reason $SU_q(2)$ is not a Kac algebra.

In all of the examples, except Example 3.1.2, we started from a Hopf $^*$-algebra and completed it with respect to some suitable C$^*$-norm. The appearance of a dense Hopf$^*$-algebra is not a coincidence. In fact, the following result holds in the general case:

**Theorem 3.1.7** *Let $(A, \Delta)$ be a compact quantum group. There exists a unique Hopf $^*$-algebra $(\mathcal{A}, \Phi)$ such that $\mathcal{A}$ is a dense unital $^*$-subalgebra of $A$ such that $\Delta(\mathcal{A}) \subseteq \mathcal{A} \odot \mathcal{A}$ and such that $\Phi$ is the restriction of $\Delta$ to $\mathcal{A}$.*

The $^*$-algebra $\mathcal{A}$ is the maximal unital $^*$-subalgebra of $A$ such that the restriction of the co-multiplication $\Delta$ to it turns it into a Hopf $^*$-algebra.

This maximal unital $^*$-subalgebra $\mathcal{A}$ is certainly contained in the unital $^*$-subalgebra $\mathcal{A}_m$ of $\mathcal{A}$ defined as the inverse image $\Delta^{-1}(A \odot A)$ of $A \odot A$ under $\Delta$. Also, it can be shown that $\Delta(\mathcal{A}_m) \subseteq \mathcal{A}_m \odot \mathcal{A}_m$. One can prove that if the Haar state is faithful on $A$, then $\mathcal{A}_m = \mathcal{A}$, so in this case $\mathcal{A}_m$ is indeed a dense Hopf $^*$-algebra.

If $A$ is commutative, then $\mathcal{A}$ consists of the linear space of the coefficient functions of the finite-dimensional unitary representations (i.e. the regular functions) of the underlying compact group. We therefore refer to $\mathcal{A}$ in Theorem 3.1.7 as the algebra of matrix coefficients. This terminology will be justified in the following subsection, where we look into the co-representation theory for compact quantum groups.

### 3.2. Finite-Dimensional Co-representations, Tannaka-Krein Duality and the Peter-Weyl Theorem

Although the proofs of some of the results in this subsection require the notion of an infinite-dimensional co-representation — in particular, the left regular co-representation — which involves the language of multiplier algebras, the results can be formulated in the setting of finite-dimensional co-representations. This is due to the quantum analogue of the classical result that states that every strongly-continuous representation of a compact group is decomposable into irreducible finite-dimensional ones.

We shall need the *leg numbering notation* to define a co-representation. Consider a unital C$^*$-algebra $A$ and a finite-dimensional Hilbert space $H$. Define unital $^*$-homomorphisms

$$A\otimes B(H)\to A\otimes A\otimes B(H):p\to p_{13} \text{ and } A\otimes B(H)\to A\otimes A\otimes B(H):p\to p_{23}$$

such that $(a \otimes x)_{13} = a \otimes 1 \otimes x$ and $(a \otimes x)_{23} = 1 \otimes a \otimes x$ for all $a \in A$, $x \in B(H)$. Notice that the tensor product $\otimes$ is just the algebraic one because $H$ is finite-dimensional.

For instructive purposes, suppose $G$ is a compact group. Let $H$ be a finite-dimensional Hilbert space. Define the linear mapping $\theta : C(G) \otimes B(H) \to C(G, B(H))$ such that $\theta(f \otimes x)(s) = f(s)x$ for all $s \in G$, $f \in C(G)$ and $x \in B(H)$. It is a $^*$-isomorphism and we shall suppress this identification in the sequel.

Let $F$ be a function in $C(G, B(H))$. A straightforward calculation shows that

$$(\Delta \otimes \iota)(F)(s,t) = F(st), \quad F_{13}(s,t) = F(s), \quad F_{23}(s,t) = F(t)$$

for all $s,t \in G$. So $F$ is multiplicative if and only if $(\Delta \otimes \iota)(F) = F_{13}F_{23}$.

Clearly, strong continuity is equivalent to norm continuity for finite-dimensional representations. Hence $U$ is a strongly continuous representation of $G$ on a finite-dimensional Hilbert space $H$ if and only if $U$ is an invertible element of the C$^*$-algebra $C(G) \otimes B(H)$ such that $(\Delta \otimes \iota)(U) = U_{13}U_{23}$.

More generally, a finite-dimensional co-representation of a compact quantum group $(A, \Delta)$ on a finite-dimensional Hilbert space $H$ is by definition an invertible element $U \in A \otimes B(H)$ such that $(\Delta \otimes \iota)(U) = U_{13}U_{23}$. We use the term 'co-representation' derived from Hopf algebra theory to distinguish these objects from 'ordinary' representations of the C$^*$-algebra $A$, which have nothing to do with representations of the possible underlying group.

The standard notions of group representation theory transfer easily to this setting:

• Let $U$,$V$ be finite-dimensional co-representations of $(A, \Delta)$ on Hilbert spaces $H$,$K$, respectively. An *intertwiner* $T$ from $U$ to $V$ is a linear mapping from $H$ to $K$ such that $V(1 \otimes T) = (1 \otimes T)U$. The set of intertwiners will be denoted by $\mathrm{Mor}(U, V)$. The co-representations $U$ and $V$ are *equivalent*, denoted $U \cong V$, if there exists an invertible intertwiner from $U$ to $V$.
• A subspace $K$ of $H$ is called *invariant* for a finite-dimensional co-representation $U$ of $(A, \Delta)$ on $H$ if $(1 \otimes P_K)U(1 \otimes P_K) = U(1 \otimes P_K)$, where $P_K$ is the orthogonal projection of $H$ onto $K$. We say that $U$ is *irreducible* if it has no non-trivial invariant subspaces.

A finite-dimensional co-representation $U$ of $(A, \Delta)$ on $H$ is called *unitary* if it is a unitary element in the unital C$^*$-algebra $A \otimes B(H)$. The following assertion, which requires a fairly straightforward argument, shows that there is no restriction in working with finite-dimensional unitary co-representations.

**Proposition 3.2.1** *Suppose $U$ is a finite-dimensional co-representation of $(A,\Delta)$ on $H$, and let $h$ be the Haar state on $(A,\Delta)$. Put $Q = (h \otimes \iota)(U^*U)$, which is an invertible positive operator on $H$, and define $V = (1 \otimes Q^{\frac{1}{2}})U(1 \otimes Q^{-\frac{1}{2}})$. Then $V$ is a finite-dimensional unitary co-representation of $(A,\Delta)$ on $H$ and $Q^{\frac{1}{2}} \in Mor(U,V)$. Thus any finite-dimensional co-representation of $(A,\Delta)$ is equivalent to a finite-dimensional unitary co-representation of $(A,\Delta)$.*

For any finite-dimensional co-representations $U \in B(H) \otimes A$ and $V \in B(K) \otimes A$ of $(A,\Delta)$, the set $\mathrm{Mor}(U,V)$ is a subspace of the vector space $B(H,K)$ of all linear operators from $H$ to $K$. Schur's lemma states:

• The co-representation $U$ is irreducible if and only if $\mathrm{Mor}(U,U) = \mathbb{C}1$.
• If $U$ and $V$ are irreducible, then

$$\mathrm{Mor}(U,V) = \begin{cases} 0 & \text{if } U \ncong V \\ \mathbb{C}F \text{ for } F \text{ invertible } \in B(H,K) & \text{if } U \cong V . \end{cases}$$

Consider an additional finite-dimensional co-representation $W$ of $(A,\Delta)$. Given intertwiners $S \in \mathrm{Mor}(U,V)$ and $T \in \mathrm{Mor}(V,W)$, the composition $T \circ S$ belongs to $\mathrm{Mor}(U,W)$. If $S$ is invertible, then $S^{-1} \in \mathrm{Mor}(V,U)$. Therefore the relation $\cong$ is indeed an equivalence relation.

If $U$ and $V$ are finite-dimensional unitary co-representations, then $S^*$ belongs to $\mathrm{Mor}(V,U)$ for $S \in \mathrm{Mor}(U,V)$. Combining this with the polar decomposition of operators between Hilbert spaces, we find that finite-dimensional unitary co-representations are equivalent if and only if they are unitarily equivalent (i.e. there exists a unitary intertwiner).

Consider finite-dimensional co-representations $U$ and $V$ of $(A,\Delta)$ on $H$ and $K$ respectively. We may form the finite-dimensional direct sum co-representation $U \oplus V$ and the finite-dimensional tensor product co-representation $U \otimes V$ of $(A,\Delta)$:

• Using the canonical embedding $(A \otimes B(H)) \oplus (A \otimes B(K)) = A \otimes (B(H) \oplus B(K)) \subseteq A \otimes B(H \oplus K)$, we regard $U \oplus V := (U,V)$

as an element in $A \otimes B(H \oplus K)$. It is easy to check that $U \oplus V$ is a finite-dimensional co-representation of $(A, \Delta)$ on $H \oplus K$.

• Putting $U \otimes V = U_{12}V_{13}$, we obtain a finite-dimensional co-representation of $(A, \Delta)$ on $H \otimes K$, where we have used the identification $B(H) \otimes B(K) = B(H \otimes K)$.

It should be noted that we are using a leg notation which differs slightly from the one introduced above. Here we look at the obvious $^*$-homomorphisms:

$$A \otimes B(H) \to A \otimes B(H) \otimes B(K) : p \to p_{12}$$

and

$$A \otimes B(K) \to A \otimes B(H) \otimes B(K) : p \to p_{13} \ .$$

If $U$ and $V$ are unitary, the co-representations $U \oplus V$ and $U \otimes V$ are unitary as well.

Notice that the identity element $1 \in A$ is a 1-dimensional unitary co-representation of $(A, \Delta)$ on $\mathbb{C}$ under the identification $A = A \otimes \mathbb{C} = A \otimes B(\mathbb{C})$. It is called the *trivial co-representation* of $(A, \Delta)$. Obviously, $U \otimes 1 \cong 1 \otimes U \cong U$ for any finite-dimensional co-representation $U$.

We have manufactured an example of a concrete tensor C$^*$-category (see [19]). Namely, it is the category $\mathrm{Rep}(A, \Delta)$ whose objects are the finite-dimensional unitary co-representations of $(A, \Delta)$. Its morphisms are the intertwiners with composition $\circ$ and $^*$-operation as prescribed. The set of morphisms between two finite-dimensional co-representations is a Banach space under the operator norm, which obviously fulfils the C$^*$-norm property. The tensor product $\otimes$ is an associative bilinear functor from $\mathrm{Rep}(A, \Delta)$ to its product category with the trivial co-representation as the unit. It commutes with the involutive contravariant $^*$-functor acting as the identity on objects and as the $^*$-operation on morphisms.

This category is concrete in the sense that the objects are essentially embedded in a category of finite-dimensional Hilbert spaces. Strict associativity of the tensor product $\otimes$ can be achieved, for example, by taking these Hilbert spaces to be Hilbert subspaces of a given (properly infinite) von Neumann algebra, so that the tensor products are defined using the (strict associative) product in the ambient von Neumann algebra.

Let $U$ be a finite-dimensional unitary co-representation of $(A, \Delta)$ on a Hilbert space $H$ and suppose that $K$ is a subspace of $H$ which is invariant under $U$. Then the orthogonal complement $K^{\perp}$ of $K$ is also invariant under $U$. Unlike the classical case, the argument for this is non-trivial in the quantum case. Denote by $U_K$ the element in $A \otimes B(K)$ obtained by restricting $U$ to $K$ and similarly, denote by $U_{K^{\perp}}$ the restriction of $U$ to $K^{\perp}$. They are both finite-dimensional unitary co-representations, and the direct sum co-representation $U_K \oplus U_{K^{\perp}}$ is equivalent to $U$. It is clear from this result that any finite-dimensional unitary co-representation can be decomposed into a finite direct sum of irreducible ones. Also, the category $\mathrm{Rep}(A, \Delta)$ has sufficient sub-objects and direct sums in the sense of [19].

Another way of seeing finite-dimensional co-representations is as matrices over $A$. Let $H$ be a finite-dimensional Hilbert space and fix an orthonormal basis $e_1, \ldots, e_n$ for $H$. Now define $\theta_{ij} \in B(H)$ by $\theta_{ij}(v) = \langle v, e_j \rangle e_i$ for all $v \in H$ and $i, j \in \{1, \ldots, n\}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $H$. Then for all $i, j, k \in \{1, \ldots, n\}$, we have

$$\theta_{ij}\theta_{kl} = \delta_{jk}\theta_{il} \qquad\qquad \theta_{ij}^* = \theta_{ji} \qquad\qquad \sum_{j=1}^{n} \theta_{ii} = 1 \ .$$

Using this system of matrix units $(\theta_{ij})_{i,j=1}^n$, we identify $A \otimes B(H)$ and $M_n(A)$. Thus if $U$ is an element of $A \otimes B(H)$, there exist unique elements $U_{ij} \in A$ $(i, j = 1, \ldots, n)$ such that $U = \sum_{i,j=1}^n U_{ij} \otimes \theta_{ij}$. Moreover, $U$ is a finite-dimensional co-representation if and only if $(U_{ij})_{i,j=1}^n$ is invertible in $M_n(A)$ and $\Delta(U_{ij}) = \sum_{k=1}^n U_{ik} \otimes U_{kj}$ for all $i, j \in \{1, \ldots, n\}$. We call the elements $U_{ij}\, (i, j = 1, \ldots, n)$ the *matrix coefficients* of $U$ with respect to the basis $e_1, \ldots, e_n$.

An invertible element $U \in M_n(A)$ satisfying $\Delta(U_{ij}) = \sum_{k=1}^n U_{ik} \otimes U_{kj}$ for all $i, j = 1, \ldots, n$ will therefore be called a *matrix co-representation* of $(A, \Delta)$ of *dimension $n$*. By the discussion above, the elements $U_{ij}$, $i, j \in \{1, ..., n\}$, are matrix coefficients of a co-representation.

A compact quantum group $(A, \Delta)$ is called a *compact matrix pseudo-group*, denoted $(A, U)$, if it has a finite-dimensional matrix co-representation $U$ such that the unital ${}^*$-algebra $\mathcal{A}$ generated by its matrix coefficients $U_{ij}$ is dense in the C$^*$-algebra $A$. The matrix co-representation $U$ is called the *fundamental co-representation* of $(A, U)$.

Recall the definition of the compact quantum group $SU_q(2)$. It is a compact matrix pseudo-group with the fundamental co-representation $U \in M_2(A_u)$ given by

$$U = \begin{pmatrix} \alpha & -q\,\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$

Let us go back to the general compact quantum group setting. Define the subspace $\mathcal{A}$ of $A$ as the linear span of the set

$$\{\, U_{ij} \mid U \text{ a matrix corepresentation of } (A, \Delta),\ 1 \le i, j \le \text{dimension} U\}$$

Let $(u^\alpha)_{\alpha \in I}$ be a complete set of pairwise inequivalent finite-dimensional irreducible unitary co-representations of $(A, \Delta)$ with dimensions $n_\alpha$ (completeness means that every finite-dimensional irreducible unitary co-representation is equivalent to one of these). By convention, we have a distinguished element in $I$ (which we denote by 0) such that $u^0$ is the trivial co-representation and thus $u^0_{11} = 1$, the unit in $\mathcal{A}$.

**Theorem 3.2.2** *The following properties hold for a compact quantum group $(A, \Delta)$:*

1. *$\mathcal{A}$ is a dense unital ${}^*$-subalgebra of $A$ with Hamel basis*
$$(\, u^\alpha_{ij} \mid \alpha \in I\,,\ i, j = 1, \ldots, n_\alpha\,)\ .$$
2. *Define the map $\Phi : \mathcal{A} \to A \otimes A$ by restricting $\Delta$ to $\mathcal{A}$. Then $(\mathcal{A}, \Phi)$ is a Hopf ${}^*$-algebra with co-multiplication $\Phi$, co-unit $\varepsilon$ and antipode $S$ uniquely determined by:*
$$\Phi(u^\alpha_{ij}) = \sum_{k=1}^{n_\alpha} u^\alpha_{ik} \otimes u^\alpha_{kj} \qquad \varepsilon(u^\alpha_{ij}) = \delta_{ij} \qquad S(u^\alpha_{ij}) = {u^\alpha_{ji}}^*$$
*for all $\alpha \in I$ and $i, j = 1, \ldots, n_\alpha$.*

With the knowledge we have acquired so far, there are only three statements in this theorem that require proofs:

1. linear independence of $( u_{ij}^{\alpha} \mid \alpha \in I \, , \, i,j = 1, \ldots, n_\alpha )$,
2. density of $\mathcal{A}$ in $A$,
3. $^*$-invariance of $\mathcal{A}$.

Assertion 1 can be proved using purely algebraic techniques (see [31]), but follows more easily from Theorem 3.2.3 stated below.

Assertion 2 requires the construction of the left (or right) regular co-representation. It is in general an infinite-dimensional unitary co-representation, and will therefore be dealt with in the next subsection where infinite-dimensional co-representations are discussed. The decomposition of the left regular co-representation into irreducible finite-dimensional unitary co-representations is what it takes to get density of $\mathcal{A}$ in $A$. As a consequence, the left regular representation contains copies of all finite-dimensional irreducible unitary co-representations (occurring with multiplicity equal to their dimensions).

In the classical case, the decomposition of an infinite-dimensional strongly continuous unitary representation of a compact group $G$ into finite-dimensional unitary representations goes as follows, [26]: First reduce to a cyclic strongly continuous unitary representation $U : G \to B(H) : s \mapsto U_s$ by Zorn's lemma. Denote by $z \in H$ the cyclic vector for $U$. Then use the Haar integral $\int : C(G) \to \mathbb{C}$ to define a new inner product $(\cdot, \cdot)$ on $H$ by

$$(x, y) = \int \langle U_s x, z \rangle \, \langle z, U_s y \rangle \, ds$$

for $x, y \in H$, where $\langle \cdot, \cdot \rangle$ is the original inner product on $H$. This yields a strictly positive operator $Q \in B(H)$ determined by $\langle Qx, y \rangle = (x, y)$ for all $x, y \in H$. Now use the Banach-Steinhaus Theorem and the Lebesgue Dominated Convergence Theorem to conclude that $Q$ is compact. So by the Hilbert-Schmidt Theorem, $Q$ has a decomposition into eigenspaces. These finite-dimensional spaces are invariant subspaces of $H$ for $U$, because $Q$ is easily seen to be an intertwiner of $U$. Thus we have obtained the desired decomposition of $U$.

The proof for the general compact quantum group case is done likewise (see Section 4). We should point out that for compact matrix pseudogroups there is no need for the left regular co-representation in order to manufacture sufficiently many finite-dimensional unitary co-representations to get density of $\mathcal{A}$ in $A$. It is implicit in the axioms (laid down by Woronowicz, where he more or less imposed a Hopf $^*$-algebra structure on $\mathcal{A}$ ), that every finite-dimensional co-representation is contained in (higher) tensor products of the fundamental co-representation $U$ and its conjugate $\overline{U}$ (see the definition of $\overline{U}$ below).

Let us enter the discussion about Assertion 3 now. Suppose that $U$ is a finite-dimensional unitary co-representation of $(A, \Delta)$ on a Hilbert space $H$ for which we fix an orthonormal basis $e_1, \ldots, e_n$. Write $U = \sum_{i,j=1}^n U_{ij} \otimes \theta_{ij}$, where $U_{ij}$, $(i, j = 1, \ldots, n)$ all belong to $\mathcal{A}$. Now define $V = \sum_{i,j}^n U_{ij}^* \otimes \theta_{ij} \in A \otimes B(H)$. Then clearly $(\Delta \otimes \iota)(V) = V_{13}V_{23}$, but it is non-trivial (and to our knowledge requires the construction of the left regular co-representation) to show that $V$ is invertible. By Proposition 3.2.1, the finite-dimensional co-representation $V$ is equivalent to a finite-dimensional unitary co-representation $\overline{U}$ of $(A, \Delta)$ on $\overline{H} = H$ (it depends on the choice of the basis, but is uniquely determined up to equivalence).

It can be shown (see [36]) that $U$ and $\overline{U}$ are conjugates in the tensor C$^*$-category $\mathrm{Rep}(A, \Delta)$ in the following sense (see [19]): There exist $R \in \mathrm{Mor}(1, \overline{U} \otimes U)$ and $\overline{R} \in \mathrm{Mor}(1, U \otimes \overline{U})$ such that

$$(\overline{R}^* \otimes 1_H)(1_H \otimes R) = 1_H \quad \text{and} \quad (R^* \otimes 1_{\overline{H}})(1_{\overline{H}} \otimes \overline{R}) = 1_{\overline{H}} \, .$$

Hence $\mathrm{Rep}(A, \Delta)$ is a concrete strict tensor C$^*$-category with conjugation.

As in the classical case, every compact quantum group is completely determined by its finite-dimensional unitary co-representations. In [36], Woronowicz proved a theorem generalizing the Tannaka-Krein Theorem for compact groups to compact quantum groups. His theorem states that every concrete (embedded) strict tensor C$^*$-category with conjugation is equivalent to

$\mathrm{Rep}(A,\Delta)$ for some compact quantum group $(A,\Delta)$, which is uniquely determined up to isomorphism on the Hopf $^*$-algebra level $(\mathcal{A},\Phi)$. The category $\mathrm{Rep}(A,\Delta)$ is *symmetric* if for any finite-dimensional unitary co-representations $U \in B(H)\otimes A$ and $V \in B(K)\otimes A$, the flip $H\otimes K \to K\otimes H$ induces an equivalence between $U\otimes V$ and $V\otimes U$. In this case $A$ has to be commutative (see [37]), so $\mathrm{Rep}(A,\Delta)$ is the category of finite-dimensional unitary representations of a compact group.

Suppose we are given an abstractly-defined strict tensor C$^*$-category $\mathcal{T}$ with conjugation (see [19]). It is not automatic that $\mathcal{T}$ can be embedded into a tensor C$^*$-category of Hilbert spaces, i.e. that there exists a faithful tensor $^*$-functor from the category $\mathcal{T}$ to a tensor C$^*$-category of Hilbert spaces. Such an embedding exists whenever $\mathcal{T}$ has a symmetry (i.e. involutive braiding). This theorem, which is due to S. Doplicher and J. E. Roberts (see [6]), requires a highly non-trivial proof. They constructed such a symmetric category in the framework of algebraic quantum field theory (where no Hilbert spaces could *a priori* be attached to the objects) to produce a compact group which could be interpreted as the gauge group associated to the net of observable algebras for the quantum field theory under consideration (see [5]). The category of finite-dimensional unitary representations of this gauge group is then equivalent to the symmetric category thus constructed.

In conformal field theory, tensor C$^*$-categories appear which are braided but not symmetric, [32]. The categories correspond to co-representations of quantum groups at root of unity, [15],[11]. It should be pointed out that the root of unity quantum groups do not fit into the C$^*$-algebraic scheme of quantum groups. In view of Woronowicz's Tannaka-Krein Theorem, an abstractly-defined strict tensor C$^*$-category with conjugation thus cannot be embedded into a tensor C$^*$-category of Hilbert spaces without further restrictions. And as the root-of-unity case shows, existence of a braiding is not sufficient. In [34] it is shown that the co-representation theory of a quantum group at root of unity gives rise to an (abstract) tensor C$^*$-category.

Tensor C$^*$-categories — including ribbon categories [14] — have proved to be a vital link between quantum field theory and quantum groups. They also seem to be a meeting point with areas such as knot theory and subfactor theory of von Neumann algebras [13]. (See [33] for construction of subfactors from quantum groups, and [23], [18] for equality of q-dimension, intrinsic dimension and Jones index.)

Let us go back to the Haar state $h$ on the compact quantum group $(A, \Delta)$ with dense Hopf $^*$-algebra $(\mathcal{A}, \Phi)$. Clearly, the restriction of $h$ to $\mathcal{A}$ is a Haar functional on $(\mathcal{A}, \Phi)$ and by Hopf $^*$-algebra theory (see [1]), it follows that $h$ is faithful on $\mathcal{A}$. Obviously, $h$ is uniquely determined by its values on the linear basis appearing in Theorem 3.1.7. A combination of the identity $\Phi(u_{ij}^\alpha) = \sum_{k=1}^{n_\alpha} u_{ik}^\alpha \otimes u_{kj}^\alpha$, the linear independence of the basis under consideration and the left invariance of $h$, yield $h(u_{11}^0) = h(1) = 1$ and $h(u_{ij}^\alpha) = 0$ for all $\alpha \in I \setminus \{0\}$ and $i, j \in \{1, \ldots, n\}$.

We are looking at a special case of the orthogonality relations for the Haar state. The Peter-Weyl Theorem states that for the commutative compact quantum group $(C(G), \Delta)$, the linear basis $(\sqrt{n_\alpha} u_{ij}^\alpha \mid \alpha \in I\,,\, i, j = 1, \ldots, n_\alpha)$ forms an orthonormal basis for $L^2(G)$. Since $h$ need not be tracial in the quantum group case, the situation is a bit more complicated. The quantum Peter-Weyl Theorem, formulated and proved by Woronowicz (see [31]), takes the form:

**Theorem 3.2.3** *For every $\alpha \in I$, there exists a unique positive invertible $n_\alpha \times n_\alpha$-matrix $F^\alpha$ over $\mathbb{C}$ with $\operatorname{Tr} F_\alpha = \operatorname{Tr} (F_\alpha)^{-1}$ such that*

$$h((u_{ip}^\beta)^* u_{jq}^\alpha) = \frac{\delta_{\alpha\beta}\, \delta_{pq}\, F_{ij}^\alpha}{M_\alpha} \quad \text{and} \quad h(u_{ip}^\beta (u_{jq}^\alpha)^*) = \frac{\delta_{\alpha\beta}\;\; \delta_{ij}\, ((F^\alpha)^{-1})_{pq}}{M_\alpha}\;,$$

*where $M_\alpha = \operatorname{Tr} F_\alpha = \operatorname{Tr} (F_\alpha)^{-1}$.*

We shall not give a complete proof but will indicate how Schur's lemma enters the argument.

Take $\alpha, \beta \in I$. Fix $i \in \{1, \ldots, n_\beta\}$ and $k \in \{1, \ldots, n_\alpha\}$ and define the $n_\beta \times n_\alpha$ matrix $\Lambda^{ik}$ over $\mathbb{C}$ with matrix elements $\Lambda_{jl}^{ik} = h((u_{ij}^\beta)^* u_{kl}^\alpha)$ for all $j \in \{1, \ldots, n_\beta\}$ and $l \in \{1, \ldots, n_\alpha\}$.

The corresponding operator from $H$ to $K$, also denoted by $\Lambda^{ik}$, is an intertwiner from $u^\alpha$ to $u^\beta$:

$$\Lambda^{ik}_{jl}\, 1 = (h \otimes \iota)\Delta((u^\beta_{ij})^* u^\alpha_{kl}) = \sum_{r=1}^{n_\beta}\sum_{s=1}^{n_\alpha} h((u^\beta_{ir})^* u^\alpha_{ks})\,(u^\beta_{rj})^* u^\alpha_{sl}$$
$$= \sum_{r=1}^{n_\beta}\sum_{s=1}^{n_\alpha} (u^\beta_{rj})^* \Lambda^{ik}_{rs} u^\alpha_{sl},$$

so $1 \otimes \Lambda^{ik} = (u^\beta)^*(1 \otimes \Lambda^{ik})u^\alpha$. By unitarity of $u^\beta$, we conclude that $\Lambda^{ik} \in \mathrm{Mor}(u^\alpha, u^\beta)$ for all $i = 1, \ldots, n_\beta$ and $k = 1, \ldots, n_\alpha$.

Schur's lemma now tells us that

$$\Lambda^{ik} = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \lambda_{ik}\, 1 \text{ for some } \lambda_{ik} \in \mathbb{C} & \text{if } \alpha = \beta\ . \end{cases}$$

When $\alpha = \beta$, the matrix $F_\alpha$ is a rescaling of $(\lambda_{ik})$ such that $\mathrm{Tr}\, F_\alpha = \mathrm{Tr}\,(F_\alpha)^{-1}$. It should be pointed out however, that it still takes some work to prove the final result from this discussion.

We denote by $L^2(h)$ the Hilbert space completion of $\mathcal{A}$ with respect to the inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{A}$ given by $\langle a, b\rangle = h(b^*a)$ for all $a, b \in \mathcal{A}$.

Notice that when $h$ is tracial (which is true in the commutative, the co-commutative and the finite-dimensional case), then $\delta_{pq}\, F^\alpha_{ij} = \delta_{ij}\,((F^\alpha)^{-1})_{qp}$ for $i, j, p, q = 1, \ldots, n_\alpha$. Thus $F^\alpha$ has to be the identity matrix on $M_\alpha(\mathbb{C})$, or equivalently, the collection $(\sqrt{M_\alpha} u^\alpha_{ij} \mid \alpha \in I\,, i, j = 1, \ldots, n_\alpha\,)$ is an orthonormal basis for $L^2(h)$.

Although the linear basis $(\,u^\alpha_{ij} \mid \alpha \in I\,, i, j = 1, \ldots, n_\alpha\,)$ is not orthogonal in general, it can be orthonormalized in a very concrete way (as opposed to the Gram-Schmidt procedure). Define for every $\alpha \in I$ the $n_\alpha \times n_\alpha$-matrix $(v^\alpha_{ij})$ over $\mathcal{A}$ such that $v^\alpha_{ij} = \sqrt{M_\alpha}\,((F_\alpha)^{-\frac{1}{2}})^t\, u^\alpha_{ij}$. A direct computation shows that $(\,v^\alpha_{ij} \mid \alpha \in I\,, i, j = 1, \ldots, n_\alpha\,)$ forms an orthonormal basis for $L^2(h)$.

The fact that $h$ is not tracial can be described by a one-parameter family of multiplicative linear functionals on $\mathcal{A}$.

Namely, define for every $z \in \mathbb{C}$ a linear functional $f_z$ on $\mathcal{A}$ by $f_z(u^\alpha_{ij}) = \big((F^\alpha)^{-z}\big)_{ji}$, where $\alpha \in I$ and $i, j \in \{1, \ldots, n_\alpha\}$. This definition makes sense as $F^\alpha$ is invertible and positive.

For $\omega, \theta \in \mathcal{A}'$ and $a \in A$, put $\omega * a * \theta = (\theta \odot \iota \odot \omega)\Phi^{(2)}(a) \in \mathcal{A}$, where $\Phi^{(2)} = (\Phi \otimes \iota)\Phi = (\iota \otimes \Phi)\Phi$.

If follows from the quantum Peter-Weyl Theorem that $h(ab) = h(b(f_1 * a * f_1))$ for all $a, b \in \mathcal{A}$. In fact, one may prove the following additional properties:

1. $f_z$ is a unital multiplicative linear functional on $\mathcal{A}$.
2. $\overline{f_z(a)} = f_{-\overline{z}}(a^*)$ and $f_z(S(a)) = f_{-z}(a)$ for all $a \in \mathcal{A}$.
3. $f_0 = \varepsilon$ and $(f_y \odot f_z)\Phi = f_{y+z}$ for all $y, z \in \mathbb{C}$.

Another property which is quite surprising is that this family implements the square of the antipode in the sense that $S^2(a) = f_{-1} * a * f_1$ for all $a \in \mathcal{A}$. This statement follows from the fact that $((F_\alpha)^{-1})^t \in \mathrm{Mor}(u^\alpha, (S^2 \odot \iota)(u^\alpha))$ for all $\alpha \in I$, which needs some more co-representation theory to prove.

We point out that the family $(f_z)_{z\in\mathbb{C}}$ of functionals is uniquely determined by the conditions mentioned above and by an analyticity condition (which is immediate from the definitions).

The following equivalences are easily checked: The Haar state $h$ is tracial if and only if $f_z = \varepsilon$ for all $z \in \mathbb{C}$, which is equivalent to $S^2 = \iota$. This last condition holds if and only if $S$ is $^*$-preserving. The fact that $h$ is tracial is also equivalent to the statement that the dual discrete quantum group is unimodular. We shall come back to discrete quantum groups in a later section.

What is lurking beneath all this is the presence of certain one-parameter groups of algebra automorphisms on $\mathcal{A}$. For instance, define a one-parameter group $(\sigma_z)_{z\in\mathbb{C}}$ of algebra automorphisms on $\mathcal{A}$ by $\sigma_z(a) = f_{iz} * a * f_{iz}$ for all $a \in \mathcal{A}$ and $z \in \mathbb{C}$. One may prove that $h$ is a KMS-state whenever it is faithful. Furthermore, the one-parameter group $(\sigma_z)_{z\in\mathbb{C}}$ is then the restriction to $\mathcal{A}$ of the modular group on the C$^*$-algebra $A$ (in the sense of Tomita-Takasaki theory) for the KMS-state $h$. These one-parameter groups play a central role in the theory of locally compact quantum groups (see Section 7 (part II)).

The following discussion indicates how closely connected areas like quantum groups, tensor categories, quantum field theories, knot theory and subfactors really are. Define a function $d : \mathrm{Rep}(A, \Delta) \to \langle 0, \infty\rangle$ by the formula $\mathrm{Tr}(f_1 \otimes \iota)u$ for all $u \in \mathrm{Rep}(A, \Delta)$. It is easy to see that the following properties hold:

- $d(1) = 1$,
- $d(u \oplus v) = d(u) + d(v)$,
- $d(u \otimes v) = d(u)d(v)$,
- $d(\overline{u}) = d(u)$,

for all $u, v \in \mathrm{Rep}(A, \Delta)$. These are properties characteristic for a dimension function. It is indeed equal to the intrinsic dimension defined on the tensor C$^*$-category $\mathrm{Rep}(A, \Delta)$, (see [23]), where its relation to the q-dimension for quantized universal enveloping algebras of Lie algebras and to the quantum dimension for ribbon categories is also established. The intrinsic dimension is defined canonically in any tensor C$^*$-category with conjugation, [19]. Longo's work (see [18]) has shown how its square root can be interpreted as the Jones index of subfactors of von Neumann algebras. One may prove that $d(u)$ is larger than the ordinary dimension of the finite-dimensional unitary co-representation $u$.

Another important consequence of the 'semi-tracial' property of the Haar state $h$ is that its left kernel $\{a \in A \mid h(a^*a) = 0\}$ is a closed two-sided $^*$-ideal of $A$. Hence we may form the quotient C$^*$-algebra $A_r = A/N_h$, where $N_h$ denotes the left kernel of $h$. Let $\theta : A \to A_r$ be the quotient map. It is easy to see that the map $(\theta \otimes \theta)\Delta$ factors through the quotient $A_r$ and that $(\theta \otimes \theta)\Delta(A_r) \subset A_r \otimes A_r$. We denote the resulting map from $A_r \to A_r \otimes A_r$ by $\Delta_r$. Clearly $(A_r, \Delta_r)$ is a compact quantum group with faithful Haar measure $h_r$ determined by $h_r\theta = h$. Its dense unital $^*$-algebra $\mathcal{A}_r$ is of course $\theta(\mathcal{A})$. Since $h$ is faithful on $\mathcal{A}$, it is clear that $\theta$ is injective on $\mathcal{A}$, so the Hopf $^*$-algebra $(\mathcal{A}_r, \Phi_r)$ is isomorphic to $(\mathcal{A}, \Phi)$. Hence, one may always reduce to compact quantum groups with faithful Haar state.

Recall now the definition of the co-commutative compact quantum groups $(C^*(G), \hat{\Delta}_u)$ and $(C^*_r(G), \hat{\Delta}_r)$, where $G$ is a discrete group. The Haar state $\hat{h}_r$ on $(C^*_r(G), \hat{\Delta}_r)$ is always faithful, whereas the Haar state $\hat{h}_u$ on $(C^*(G), \hat{\Delta}_u)$ is faithful if and only if $G$ is amenable. We defined a unital $^*$-isomorphism $\pi_u$ from $C^*(G)$ to $C^*_r(G)$ with the property that $\hat{h}_u = \hat{h}_r\pi_u$. Thus $N_{\hat{h}_u} = \ker \pi_u$ and $\theta = \pi_u$. Therefore, with $(A, \Delta) = (C^*(G), \hat{\Delta}_u)$, we get $(A_r, \Delta_r) = (C^*_r(G), \hat{\Delta}_r)$. Here $\mathcal{A}_r = \mathcal{A} = \mathbb{C}[G]$.

The dual space $\mathcal{A}'$ of the Hopf $*$-algebra $(\mathcal{A}, \Phi)$ consisting of all linear functionals on $\mathcal{A}$, is a unital $^*$-algebra with product and $^*$-operation defined by $\omega\eta(a) = (\omega\otimes\eta)\Phi(a)$ and $\omega^*(a) = \overline{\omega(S(a)^*)}$ for all $\omega, \eta \in \mathcal{A}'$ and $a \in \mathcal{A}$. The unit of $\mathcal{A}$ is the co-unit $\varepsilon$ of $(\mathcal{A}, \Phi)$. Since the inclusion $\mathcal{A}' \odot \mathcal{A}' \subseteq (\mathcal{A} \odot \mathcal{A})'$ is surjective if, and only if, $\mathcal{A}$ is finite-dimensional, there is no hope that the formula $\hat{\Delta}(\omega)(x \otimes y) = \omega(xy)$ for all $\omega \in \mathcal{A}'$ and $x, y \in \mathcal{A}$, which defines an element $\hat{\Delta}(\omega) \in (\mathcal{A} \odot \mathcal{A})'$, gives a co-multiplication $\hat{\Delta}(\omega)$. Indeed, it follows from the quantum Peter–Weyl Theorem that $\Delta(h) \in \mathcal{A}' \odot \mathcal{A}'$ if and only if $\mathcal{A}$ is finite-dimensional, where $h$ is the Haar state on $(\mathcal{A}, \Phi)$.

Let $U$ be a finite-dimensional unitary co-representation of $(A, \Delta)$ on a Hilbert space $H$. The formula $(\omega \odot \iota)(U)$ makes sense for any functional $\omega \in \mathcal{A}'$, and it can be shown that the mapping $\pi_U : \mathcal{A}' \to B(H) : \omega \mapsto (\omega \otimes \iota)(U)$ is a finite-dimensional weak $^*$-continuous unital $^*$-representation of $\mathcal{A}'$ on $H$. The correspondence $U \mapsto \pi_U$ is a bijection between finite-dimensional unitary co-representations of $(\mathcal{A}, \Delta)$ and finite-dimensional weak $^*$-continuous unital $^*$-representations of $\mathcal{A}'$. All the notions concerning the co-representation theory of $(A, \Delta)$ and the representation theory of $\mathcal{A}'$ (intertwiners, irreducibility, ...) transform naturally under this bijection.

Suppose that the compact quantum group $(A, \Delta)$ is co-commutative and let $U$ be a finite-dimensional irreducible unitary co-representation of $(A, \Delta)$. Then $\mathcal{A}'$ is commutative and therefore the irreducible representation $\pi_U$ has to be 1-dimensional. This implies that the matrix co-representation associated to $U$ is nothing but a unitary element $u$ in $\mathcal{A}$ such that $\Delta(u) = u \otimes u$. Define $G$ to be the subgroup of the unitary group of $\mathcal{A}$ consisting of all group–like elements. It is now easy to see that the Hopf $^*$-algebras $(\mathcal{A}, \Phi)$ and $(\mathbb{C}[G], \hat{\Delta})$ are isomorphic. If $h$ is faithful, then $(A, \Delta)$ is thus isomorphic to $(C^*_r(G), \hat{\Delta}_r)$. Of course, the quantum Peter-Weyl Theorem is a triviality here because $h(\delta_s^*\delta_t) = h(\delta_{s^{-1}t})$, which by definition is 1 if $s = t$ and 0 otherwise.

## 4. Left Regular Co-representations and Multiplicative Unitaries

We shall need multiplier algebras in order to formulate the notion of infinite-dimensional co-representations of compact quantum groups. They are also an indispensable tool for the study of non-compact quantum groups and multiplier Hopf $^*$-algebras. For this reason, we give a slightly more general definition of a multiplier algebra than is customary.

**Definition 4.1** *Consider a $^*$-algebra $A$ satisfying the Frobenius property. Denote by* $\mathrm{End}(A)$ *the unital algebra of linear maps from $A$ to $A$. Define the set $M(A)$ to be*

$$\{\, \mathrm{T} \in \mathrm{End(A)} \mid \exists \mathrm{S} \in \mathrm{End(A)} \text{ such that } \mathrm{T(a)^*b = a^*S(b)}\ \forall \mathrm{a, b} \in \mathrm{A}\,\}\ .$$

*Then $M(A)$ is a unital subalgebra of* $\mathrm{End}(A)$*. The linear map $S$ associated to a given $T \in M(A)$ is uniquely determined by the Frobenius property and we denote it by $T^*$. $M(A)$ is a unital $^*$-algebra with $T \mapsto T^*$ as $^*$-operation, .*

For $a \in A$, define $L_a \in M(A)$ by $L_a(b) = ab$ for all $b \in A$. By the Frobenius property, the map $A \to M(A)$, $a \mapsto L_a$, is an injective $^*$-homomorphism, and its image, which we shall identify with $A$, is a two-sided $^*$-ideal in $M(A)$. Moreover, the ideal $A$ is essential in the following sense: An element $x \in M(A)$ satisfying $xa = 0$ for all $a \in A$, has to be equal to zero. It is clear that $M(A) = A$ if, and only if, $A$ is unital.

Let $A$, $B$ be two $^*$-algebras having the Frobenius property. The formula $(S \otimes T)(a \otimes b) = S(a) \otimes T(b)$ for all $a \in A$, $b \in B$, $S \in M(A)$ and $T \in M(B)$ defines an embedding of $M(A) \otimes M(B)$ into $M(A \otimes B)$. In general it is not surjective.

When $A$ is a C$^*$-algebra, the Closed Graph Theorem implies that $M(A)$ consists of bounded operators. Also, $M(A)$ is a unital C$^*$-algebra with the operator norm. We give two basic examples of multiplier algebras:

• Let $X$ be a locally compact, Hausdorff space. Denote by $C_0(X)$ the C$^*$-algebra of continuous functions on $X$ that vanish at infinity. Then $M(C_0(G))$ is the C$^*$-algebra of all bounded continuous functions on $X$. Hence $M(C_0(G))$ is $^*$-isomorphic to $C(\tilde{X})$, where $\tilde{X}$ is the Stone-Cech compactification of $X$.
• Let $B_0(H)$ be the C$^*$-algebra of compact operators on a Hilbert space $H$. Then $M(B_0(H))$ is $^*$-isomorphic to $B(H)$.

Suppose we are given two C$^*$-algebras $A$ and $B$ and a $^*$-homomorphism $\pi : A \to M(B)$. We call $\pi$ *non-degenerate* if the linear span of the set $\{\pi(a)b \mid a \in A, b \in B\}$ is dense in $B$. It is possible to show that every non-degenerate $^*$-homomorphism $\pi : A \to M(B)$ has a unique extension to a unital $^*$-homomorphism $\overline{\pi} : M(A) \to M(B)$. We denote $\overline{\pi}$ by the same symbol $\pi$ in the sequel.

Again we need the leg numbering notation. In defining it, we have to be more cautious in the present setting. Take three C$^*$-algebras $A$, $B$, $C$. It can be shown that there exists a non-degenerate $^*$-homomorphism $\theta_{13} : A \otimes C \to M(A \otimes B \otimes C)$ such that $\theta_{13}(a \otimes c) = a \otimes 1 \otimes c$ for all $a \in A$, $c \in C$. Thus, it has a unique extension to $M(A \otimes C)$. Set $x_{13} = \theta_{13}(x)$ for all $x \in M(A \otimes C)$. The other variants of the leg numbering notation are defined similarly.

Take two C$^*$-algebras $A, B$ and $x \in M(A \otimes B)$. For $\omega \in A^*$, the element $(\omega \otimes \iota)(x)$ of $M(B)$ is defined in the following way:

It can be shown (see [27]) that $\omega \odot \iota : A \odot B \to B$ has a unique extension to a continuous linear map $\omega \otimes \iota : A \otimes B \to B$. The next step is to extend the map $\omega \otimes \iota$ to $M(A \otimes B)$. It can be shown (see [20]) that $\omega \otimes \iota$ has a unique bounded linear extension $\omega \overline{\otimes} \iota : M(A \otimes B) \to M(B)$ such that $b\,(\omega\overline{\otimes}\iota)(X) = (\omega\overline{\otimes}\iota)((1 \otimes b)X)$ and $(\omega\overline{\otimes}\iota)(X)\,b = (\omega\overline{\otimes}\iota)(X(1 \otimes b))$ for all $b$ in $M(A)$ and $X$ in $M(A \otimes B)$. Now put $(\omega \otimes \iota)(x) := (\omega\overline{\otimes}\iota)(x)$ for $x \in M(A \otimes B)$. Of course, a similar construction produces the element $(\iota \otimes \theta)(x)$ of $M(A)$ for any $\theta \in B^*$ and $x \in M(A \otimes B)$.

Suppose $G$ is a compact group and consider a map $U$ from $G$ to $B(H)$, where $H$ is a (not necessarily finite-dimensional) Hilbert space. Identify $C(G, B_0(H))$ with $C(G) \otimes B_0(H)$. Define a linear map $\tilde{U}$ from $C(G) \otimes B_0(H)$ to the set of all functions from $G$ to $B_0(H)$ by $\tilde{U}(F)(s) = U_s F(s)$ for all $F \in C(G, B_0(H))$ and $s \in G$. Then $U$ is bounded and strongly $^*$-continuous if and only if $\tilde{U} \in M(C(G) \otimes B_0(H))$. Thus arguing as in the finite-dimensional case, we see that $U$ is a strongly-continuous unitary representation of $G$ on $H$ if, and only if, $\tilde{U}$ is a unitary element of the multiplier C$^*$-algebra $M(C(G) \otimes B_0(H))$ and

$$(\Delta \otimes \iota)\tilde{U} = \tilde{U}_{13}\tilde{U}_{23}\,.$$

Here we have extended the non-degenerate $^*$-homomorphism $\Delta \otimes \iota$ to the multiplier algebra $M(C(G) \otimes B_0(H))$ as explained above.

Notice that requiring $\tilde{U}$ to belong to $M(C(G)) \otimes M(B_0(H))$ amounts to requiring that $U$ be norm-continuous, which in general is a too strong condition. For instance, the left regular representation is norm-continuous if, and only if, it is represented on a finite-dimensional Hilbert space.

**Definition 4.2** *Let $(A, \Delta)$ be a compact quantum group and $H$ a Hilbert space. A unitary element $U \in M(A \otimes B_0(H))$ is called a unitary co-representation of $(A, \Delta)$ if $(\Delta \otimes \iota)(U) = U_{13}U_{23}$.*

We will not look at the tensor category of all (infinite-dimensional) unitary co-representations of a compact quantum group, but will just mention some results relating to finite-dimensional unitary co-representations and the Hopf $^*$-algebra $(\mathcal{A}, \Phi)$.

However, let us see how intertwiners and invariant subspaces are defined in the infinite-dimensional setting:

- Let $U$ and $V$ be unitary co-representations of $(A, \Delta)$ on Hilbert spaces $H$ and $K$, respectively, and suppose $T \in B(H, K)$. We say that $T$ is an *intertwiner* from $U$ to $V$ if $(\omega \otimes \iota)(V)\, T = T\, (\omega \otimes \iota)(U)$ for all $\omega \in A^*$
- A closed subspace $K$ of $H$ is said to be *invariant* under $U$ if $(\omega \otimes \iota)(U)\, K \subseteq K$ for all $\omega \in A^*$.

The theorem below implies that the orthogonal complement $K^\perp$ of $K$ is invariant under $U$ if $K$ is. As a consequence, the restrictions of $U$ to $K$ and $K^\perp$ are unitary co-representations with direct sum equivalent to $U$ (i.e. there exists a unitary intertwiner between them).

The terminology (irreducibility, tensor products,...) used for finite-dimensional unitary co-representations is now easily generalized to infinite-dimensional unitary co-representations.

Let $U$ be a unitary co-representation of $(A, \Delta)$ on Hilbert spaces $H$. Define the subspace $B$ of $B(H)$ to be the closure of the set $\{ (ha \otimes \iota)U^* \mid a \in A \}$. The importance of $B$ is revealed in the following theorem:

**Theorem 4.3** *Let the notation be as above. The following properties hold:*

- *$B$ is a non-degenerate $C^*$-subalgebra of $B(H)$,*
- *$U \in M(A \otimes B)$,*
- *$(1 \otimes T)U = U(1 \otimes T) \Leftrightarrow T \in B'$ for all $T \in B(H)$ .*

Here $B'$ is the commutant of $B$ in $B(H)$, and so it is a von Neumann algebra. Clearly, we have regarded $1 \otimes T$ for $T \in B(H)$ as an element in $M(A \otimes B_0(H))$. It can be shown that $T$ intertwines $U$ with itself if, and only if, $U$ satisfies the last condition in the theorem. From this and the $^*$-invariant property of $B$, it follows that orthogonal complements of $U$-invariant subspaces are $U$-invariant.

**Theorem 4.4** *Every irreducible unitary co-representation of a compact quantum group $(\mathcal{A}, \Delta)$ is finite-dimensional. Any unitary co-representation $U$ of $(A, \Delta)$ on a Hilbert space $H$ can be decomposed into a direct sum of finite-dimensional unitary co-representations. More precisely, there exists a family of mutually orthogonal finite-dimensional subspaces $(H_i)_{i \in I}$ of $H$ such that $H = \oplus_{i \in I} H_i$ and each $H_i$ is invariant under $U$ and the restriction $U_i$ of $U$ to $H_i$ is a finite-dimensional unitary co-representation of $(\mathcal{A}, \Delta)$. In this case we write $U = \oplus_{i \in I} U_i$.*

We sketch the proof (see [31] for more details). The first statement is immediate from the second one. Define for $v \in H$, the rank-one operator $\theta_{v,v}$ on $H$ by $\theta_{v,v}(w) = \langle w, v \rangle v$ for $w \in H$. Next, put $Q_v = (h \otimes \iota)(U^*(1 \otimes \theta_{v,v})U)$, where $h$ is the Haar state on $(A, \Delta)$. The element $U^*(1 \otimes \theta_{v,v})U$ obviously belongs to $A \otimes B_0(H)$

(remember that $A$ is unital), so $Q_v$ is a positive compact operator on $H$. A straightforward calculation shows that $U$ and $1 \otimes Q_v$ commute in $M(A \otimes B_0(H))$.

By taking an orthonormal basis $(e_i)_{i \in I}$, we get a family of rank-one projections $(\theta_{e_i,e_i})_{i \in I}$ which sum up to the identity operator on $H$ in the strong topology. Using strict continuity arguments, one sees that the family $(Q_{e_i})_{i \in I}$ is strongly summable and that the sum equals the identity operator on $H$. In particular $Q_{e_j} \neq 0$ for some $j \in I$. Since $Q_{e_j}$ is compact and non-zero, it has a finite-dimensional eigenspace (corresponding to any strictly positive eigenvalue). Clearly, this eigenspace is invariant under $U$. The restriction of $U$ to the orthogonal complement is again a unitary co-representation (here this is obvious). Applying Zorn's lemma it is not difficult to see that we get the desired orthogonal decomposition of $H$ into finite-dimensional $U$-invariant subspaces. Using finite-dimensional co-representation theory, any unitary co-representation can therefore be decomposed into a direct sum of finite-dimensional irreducible unitary co-representations.

We now proceed to define the most important unitary co-representation, the left regular one, which incorporates the co-product of the quantum group. Let $(H, \pi, \Omega)$ be a GNS-representation for the Haar state $h$, i.e. $H$ is a Hilbert space, $\pi : A \to B(H)$ is a unital $^*$-homomorphism and $\Omega$ is an element in $H$ such that $\overline{\pi(A)\Omega} = H$ and $h(a) = \langle \pi(a)\Omega, \Omega \rangle$ for all $a \in A$.

Pick a faithful unital $^*$-representation $\theta$ of $A$ on a Hilbert space $K$. Let $a_1, \ldots, a_n \in A$ and $v_1, \ldots, v_n \in K$. Then the left invariance of $h$ implies that

$$\begin{aligned} \| \textstyle\sum_{\mathrm{i=1}}^{\mathrm{n}} (\theta \otimes \pi)(\Delta(\mathrm{a_i}))(\mathrm{v_i} \otimes \Omega) \|^2 &= \textstyle\sum_{\mathrm{i,j=1}}^{\mathrm{n}} (\omega_{\mathrm{v_i,v_j}} \otimes \varphi)(\Delta(\mathrm{a_j^* a_i})) \\ &= \textstyle\sum_{i=1}^{n} \langle v_i, v_j \rangle \, \langle \pi(a_i)\Omega, \pi(a_j)\Omega \rangle = \| \textstyle\sum_{i=1}^{n} v_i \otimes \pi(a_i)\Omega \|^2. \end{aligned}$$

From this we conclude that we have a well-defined isometry $U \in B(K \otimes H)$ such that $U(v \otimes \pi(a)\Omega) = (\theta \otimes \pi)(\Delta(a))(v \otimes \Omega)$ for all $a \in A$ and $v \in K$. The density of $\Delta(A)(A \otimes 1)$ in $A \otimes A$ implies that $U$ has dense range and is therefore unitary. In fact, the following proposition holds.

**Proposition 4.5** *There exists a unique unitary element $V$ of $M(A \otimes B_0(H))$ such that $V^*(v \otimes \pi(a)\Omega) = (\theta \otimes \pi)(\Delta(a))(v \otimes \Omega)$*

*for all $a \in A$ and $v \in K$. Moreover, the element $V$ is a unitary co-representation of $(A, \Delta)$.*

**Proof:** The formula

$$V^*(1 \otimes \theta_{\pi(a)\Omega,v}) = (\theta \otimes \pi)(\Delta(a))(\Delta(a))(1 \otimes \theta_{\Omega,v})$$

for all $a \in A$ and $v \in K$, implies that $V^*(A \otimes B_0(H)) = A \otimes B_0(H)$. Invoking the unitarity of $V$, we get $V(A \otimes B_0(H)) = A \otimes B_0(H)$. So we can regard $V$ as an element in $\mathrm{End}(A \otimes B_0(H))$ and it is easy to see that its adjoint in the sense of Definition 4.1 is $V^*$, regarded as an element in $\mathrm{End}(A \otimes B_0(H))$. Consequently, $V$ can be considered as an element in $M(A \otimes B_0(H))$.

The co-representation property follows from the identity

$$(\omega \otimes \iota)(V^*)\, \pi(a)\Omega = \pi((\omega \otimes \iota)\Delta(a))\Omega$$

for all $\omega \in A^*$, combined with the coassociativity of $\Delta$. □

The co-representation $V$ is that which is called the *left regular co-representation* of the compact quantum group $(A, \Delta)$. We have, for all $a, b \in A$, that

$$(\iota \otimes \omega_{\pi(a)\Omega,\pi(b)\Omega})(V) = (\iota \otimes h)(\Delta(b^*)(1 \otimes a)) \ .$$

Now decompose $V$ according to Theorem 4.4 into a direct sum $\oplus_{i \in I} V_i$ of finite-dimensional co-representations of $(A, \Delta)$. Clearly, $(\iota \otimes \omega_{v_i,w_i})(V_i) \in \mathcal{A}$ for all $i \in I$ and $v_i, w_i \in H_i$. Therefore $(\iota \otimes \omega_{\pi(a)\Omega,\pi(b)\Omega})(V)$ and thus $(\iota \otimes h)(\Delta(b^*)(1 \otimes a))$ belongs to the closure of $\mathcal{A}$ for all $a, b \in A$. Since $\{\, (\iota \otimes h)(\Delta(b^*)(1 \otimes a)) \mid a, b \in A \,\}$ is a dense subset of $A$, we conclude that $\mathcal{A}$ is dense in $A$, which proves Theorem 3.1.7.

**Definition 4.6** *Define the unitary element $W \in B(H \otimes H)$ by $W = (\pi \otimes \iota)(V)$. Then $W^*(v \otimes \pi(a)\Omega) = (\pi \otimes \pi)(\Delta(a))(v \otimes \Omega)$ for all $a \in A$ and $v \in H$. The operator $W$ is called the multiplicative unitary of the compact quantum group $(A, \Delta)$. It satisfies the pentagonal equation: $W_{12} W_{13} W_{23} = W_{23} W_{12}$ (this is a consequence of the co-associativity of $\Delta$).*

The following properties hold:

- $\pi(A)$ is the closure of the set $\{(\iota\otimes\omega)(W) \mid \omega\in B_0(H)\}$ in $B(H)$.
- $(\pi\otimes\pi)\Delta(a) = W^*(1\otimes\pi(a))W$ for all $a\in A$.

Hence the co-multiplication is essentially encoded in $W$.

These considerations lead to the work of Baaj & Skandalis [3], who study multiplicative unitaries in their own right (in that they are not necessarily constructed from a prescribed quantum group). These authors construct quantum-group-like objects with co-multiplication from multiplicative unitaries.

**Definition 4.7** *Consider a Hilbert space $H$ and a unitary element $W\in B(H\otimes H)$ satisfying the pentagonal equation (so $W_{12}W_{13}W_{23} = W_{23}W_{12}$). We call $W$ a multiplicative unitary on $H$.*

As in the case of a compact quantum group, one introduces:

1. The closed subspaces $A=[(\iota\otimes\omega)(W) \mid \omega\in B_0(H)^*]$ and $\hat{A} = [(\omega\otimes\iota)(W) \mid \omega\in B_0(H)^*]$ of $B(H)$.
2. Linear maps $\Delta : A\to B(H\otimes H)$ and $\hat{\Delta} : \hat{A}\to B(H\otimes H)$ given by $\Delta(x) = W^*(1\otimes x)W$ for all $x\in A$ and $\hat{\Delta}(x) = W(x\otimes 1)W^*$ for all $x\in\hat{A}$.

Here $[\ ]$ denotes the closed linear span of the elements under consideration.

Baaj and Skandalis formulated a certain regularity condition for multiplicative unitaries and proved that if $W$ satisfies this regularity condition, then:

- $A$ and $\hat{A}$ are non-degenerate C$^*$-subalgebras of $B(H)$,
- $(\Delta\otimes\iota)\Delta = (\iota\otimes\Delta)\Delta \quad$ and $\quad (\hat{\Delta}\otimes\iota)\hat{\Delta} = (\iota\otimes\hat{\Delta})\hat{\Delta}$,
- $\Delta(A)(A\otimes 1)$ and $\Delta(A)(1\otimes A)$ are dense subspaces of $A\otimes A$,
- $\hat{\Delta}(\hat{A})(\hat{A}\otimes 1)$ and $\hat{\Delta}(\hat{A})(1\otimes\hat{A})$ are dense subspaces of $\hat{A}\otimes\hat{A}$.

However, the regularity condition turned out to be not very suitable for the general framework of quantum groups. Baaj himself pointed out (see [2]) that the multiplicative unitary associated to the quantum group $E(2)$ does not satisfy the regularity condition.

Later Woronowicz (see [35]) introduced the notion of *manageability* for multiplicative unitaries. This probably covers the general case. Assuming this condition, he was able to prove the same properties for $(A, \Delta)$ and $(\hat{A}, \hat{\Delta})$ as Baaj and Skandalis proved using their regularity condition. Woronowicz also constructed an antipode-like object that admitted a polar decomposition (under the assumption of his manageability condition).

The objects $(A, \Delta)$ and $(\hat{A}, \hat{\Delta})$ are to be thought of as dual to each other. For instance, one of them is a compact quantum group if and only if the other one is a discrete quantum group.

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