A SPECTRAL PROPERTY FOR
THE SYMBOLIC DYNAMICAL SYSTEM

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Let \((X, \phi)\) be a topological dynamical system, that is, \(X\) is a compact Hausdorff space and \(\phi : X \to X\) is a homeomorphism. Recently, a great deal of attention has been paid to topological dynamical systems with the so-called chaotic behaviour, see [4, 6] and the bibliography therein for a comprehensive discussion about what the chaotic behaviour means mathematically.

We can consider the linear operator \(T_\phi\) defined by \(T_\phi(f) = f \circ \phi\), where \(f\) is a function on \(X\). If \((X, \phi)\) and \((Y, \psi)\) are topological dynamical systems, then they are isomorphic and we write \(\phi \cong \psi\), if there exists a homeomorphism \(h : X \to Y\) such that \(\phi = h^{-1} \psi h\). Clearly, \(T_\phi = T_{h \circ \phi} T_{h^{-1}}\), and therefore the spectral properties of \(T_\phi\) are natural invariants with respect to the isomorphism relationship. In particular, if the spectral properties of \(T_\phi\) are distinct from those of \(T_{\psi}\), then we have \(\phi \not\cong \psi\). It is interesting to investigate spectral properties of the operators \(T_\phi\) that are typical for the dynamical systems \((X, \phi)\) with chaotic behaviour and this is the principal topic of the paper.

Consider the two-element group \(\mathbb{Z}_2 = \{0, 1\}\) under addition modulo 2 and define the set \(\Omega = \mathbb{Z}_2^\mathbb{N}\); that is, \(\Omega\) is the set of all doubly infinite sequences \(\omega = (\omega_n)_{n \in \mathbb{Z}}\) with \(\omega_n \in \mathbb{Z}_2\). Obviously \(\Omega\) is compact when endowed with the product topology and operation. Define the shift transformation \(\phi : \Omega \to \Omega\) by setting \(\phi(\omega) = (\omega')\), where \(\omega'_n = \omega_{n+1}\) for all \(n \in \mathbb{Z}\). The dynamical system \((\Omega, \phi)\) is called the symbolic dynamical system. This is a traditional first choice for the system exemplifying chaotic behaviour [3]. Moreover, sometimes a dynamical system \((X, \psi)\) is said to have chaotic behavior if there exists an invariant subset.
$Y \subseteq X$ such that the dynamical system $(Y, \psi|_{Y})$ is isomorphic to the dynamical system $(\Omega, \phi)$ [4].

**Theorem 1.**
(a) There exists a continuous function $g_0 \in C(\Omega)$ such that for each complex $\lambda, |\lambda| = 1$ the open ball $B(g_0, 1/8)$ does not intersect the range of $T_\phi - \lambda I$.
(b) 1 is the only eigenvalue of $T_\phi$.

The property (a) of the operator $T_\phi$ from Theorem 1 contrasts with the properties of the operator $T_\phi$ in $L_2(\mu)$ with the Bernoulli invariant measure $\mu$. [5]: for each $\lambda \in \mathbf{T}$ the range of the operator $\lambda I - T_\phi$ is dense in $L_2(\mu)$. Note also that the property mentioned in the theorem is invariant with respect to isomorphism:

**Corollary 2.** Let $(Y, \psi)$ be a topological dynamical system that is isomorphic to $(\Omega, \phi)$. Then the following spectral properties are valid:
(a) there exists a continuous function $g_0 \in C(Y)$ such that for each $\lambda \in \mathbf{T}$, the open ball $B(g_0, 1/8)$ does not intersect the range of $T_\psi - \lambda I$;
(b) 1 is the only eigenvalue of $T_\psi$.

The combination of spectral properties mentioned in Theorem 1 looks rather peculiar and may be typical only for dynamical systems which behave in a similar way to the hyperbolic homeomorphisms, [3]. This suggests that the following definition could be useful:

**Definition.** A topological dynamical system $(X, \psi)$ is said to be $s$-chaotic if the operator $T_\psi : C(X) \to C(X)$ has no eigenvalues apart from 1 and, on the other hand, there exists an open ball $B(g_0, \varepsilon) \subseteq C(X)$ satisfying

$$B(g_0, \varepsilon) \cap \bigcup_{\lambda \in \mathbf{T}} (T_\psi - \lambda I)(C(X))) = \emptyset.$$

Note that by Corollary 2 this definition is in line with the definition of chaotic behaviour as suggested in [4]. The tentative analysis shows that this definition does not apply to the dynamical systems.
(Y, ψ) which are traditionally considered as ‘non-chaotic’ or have a non-chaotic component.

**Example 3.** As an example consider the irrational rotation ψ_α of a circle:

\[ e^{iθ} \mapsto e^{i(θ + 2πα)} \]

where \( 0 \leq θ < 2π \) and \( α \) is a fixed irrational real number. This mapping is a typical example of an ergodic but not chaotic mapping. The spectrum of the operator \( T_{ψ_α} \) coincides with the unit circle of the complex plane, but its further properties contrast sharply with those mentioned in the definition of s-chaotic behaviour: the operator \( T_{ψ_α} \) has a countable number of eigenvalues \( λ_k = e^{2πiαk}, k = 0, \pm 1, \pm 2, \ldots \) and for any \( λ \neq λ_k, k = 0, \pm 1, \pm 2, \ldots \) the range of the operator \( T_{ψ_α} − λI \) is dense in \( C(T) \).

On the other hand, the chaotic homeomorphisms different from the symbolic dynamical system are likely to be s-chaotic.

**Problem.** Prove that an algebraic toral automorphism \([4]\) is s-chaotic.

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