ON A COMMENT OF DOUGLAS
CONCERNING WIDOM’S THEOREM

Mícheál Ó Searcóid

§1 Introduction.
Douglas, after presenting an adaptation of Widom’s proof [2] that every Toeplitz operator has connected spectrum, comments, “Despite the elegance of the preceding proof of connectedness, we view it as not completely satisfactory for two reasons: First, the proof gives us no hint as to why the result is true. Second, the proof seems to depend on showing that the set of some kind of singularities for a function of two complex variables is connected ...” [1, p.196]. The purpose of this note is to demonstrate that one of the variables referred to by Douglas can be effectively suppressed by extensive use of the F. & M. Riesz Theorem; the modified proof is, I believe, somewhat cleaner.

§2 Preliminary concepts.
The items in this section are well-known and are covered in [1].

Notation. The unit circle is denoted by $T$. We consider the spaces $L^p = L^p(T)$ for $p = 1, 2, \infty$, where the measure is Lebesgue measure and the vectors are treated as functions defined almost everywhere. The functions $e_n : z \mapsto z^n$ ($n \in \mathbb{Z}$) form an orthonormal basis for the Hilbert space $L^2$. The Hardy spaces $H^p$ are $H^p = \{ f \in L^p : \int_T f e_n = 0 \ \forall n > 0 \}$, ($p = 1, 2, \infty$). $P$ will denote the orthogonal projection from $L^2$ onto $H^2$. Note that $L^\infty$ and $H^\infty$ are Banach algebras, that $L^\infty \subset L^2 \subset L^1$ and $H^\infty \subset H^2 \subset H^1$ and that $L^\infty L^2 = L^2$. For $\phi \in L^\infty$, $\sigma(\phi)$ will denote the spectrum. $\{ \lambda \in \mathbb{C} : \phi - \lambda \text{ not invertible in } L^\infty \}$, of $\phi$ in $L^\infty$; note that this is the same as the essential range of $\phi$, namely, the set of all $\lambda \in \mathbb{C}$.
such that, for every $\epsilon > 0$, the set \{ $z \in T : |\phi(z) - \lambda| < \epsilon$ \} has positive measure. For $T \in \mathcal{B}(H^2)$, the algebra of bounded linear operators on $H^2$, $\sigma(T)$ will denote the spectrum of $T$ in $\mathcal{B}(H^2)$.

**Definition.** For each $\phi \in L^\infty$ we define the Toeplitz operator $T_\phi \in \mathcal{B}(H^2)$ by $T_\phi f = P(\phi f)$ for each $f \in H^2$.

**Proposition 1.** Suppose $f \in L^1$ and $\int_T f e_n = 0$ for all $n \in \mathbb{Z}$. Then $f = 0$.

**Proposition 2.** Suppose $f, g \in H^2$. Then $fg \in H^1$.

**Proposition 3.** Suppose $\phi \in L^\infty$. Then $T_\phi^* = (T_\phi)^*$.

**Proposition 4.** Suppose $\phi \in L^\infty$. Then $\sigma(\phi) \subseteq \sigma(T_\phi)$. (This implies, of course, that, if $T_\phi$ is invertible, then so is $\phi$. However, it is worth noting that the inverse of $T_\phi$ is not, except in very special cases, equal to $T_{\phi^{-1}}$.)

**F. & M. Riesz Theorem.** Suppose $f \in H^2$. If $f \neq 0$ then the set of zeroes of $f$ has zero measure. (It follows immediately from this that if $\phi \in H^\infty$ and the essential range of $\phi$ is countable, then $\phi$ is essentially constant.)

§3 The connectedness.

**Proposition 5.** Suppose $\Gamma$ is a simple closed integration path and $K$ is a compact subset of the complex plane with $K \cap \Gamma = \emptyset$. Let $\phi \in L^\infty$ be such that $\sigma(\phi) = K$. Then $\Gamma$ fails to separate $K$ if and only if

$$\exp \left( P \int_{\Gamma} \frac{d\mu}{\phi - \mu} \right) = e_0.$$ 

**Proof:** $\Gamma$ fails to separate $\sigma(\phi)$ if and only if the winding number function in $L^\infty$

$$w(\Gamma, \phi) = \frac{1}{2\pi i} \int_{\Gamma} \frac{d\mu}{\mu - \phi}$$

is (essentially) constant. Since $w(\Gamma, \phi)$ has only integer values, the F. & M. Riesz Theorem ensures that this happens if and only if $w(\Gamma, \phi)$ is in $H^\infty$, i.e., if and only if

$$P \int_{\Gamma} \frac{d\mu}{\mu - \phi} = \int_{\Gamma} \frac{d\mu}{\mu - \phi}.$$
Since \( \exp \left( \int_{\Gamma} \frac{d\mu}{\phi - \mu} \right) = e_0 \), the result follows by invoking the F. & M. Riesz Theorem again.

**Proposition 6.** Suppose \( \phi \in L^\infty \) and \( T_\phi \) is invertible in \( B(H^2) \).
If \( f \in H^2 \) satisfies \( T_\phi f = e_0 \), then \( f^{-1} \in H^2 \) and \( T_{\phi^{-1}} f^{-1} = e_0 \).

**Proof:** Firstly, since \( T_\phi \) is invertible, Propositions 3 and 4 ensure that \( \phi \) and \( \bar{\phi} \) are invertible in \( L^\infty \) and that \( T_{\phi^{-1}} = T_{\bar{\phi}} \) is invertible in \( B(H^2) \). In particular, there is exactly one vector mapped to \( e_0 \) by \( T_{\phi^{-1}} \), so that \( \dim(\overline{\phi H^2} \cap \overline{H^2}) = 1 \). It follows that the space \( H^2 \cap \overline{\phi^{-1}H^2} \) has dimension 1 and then also that
\[
\dim(\phi^{-1}H^2 \cap \overline{H^2}) = 1.
\]

We deduce that \( T_{\phi^{-1}} \) is injective and that there exists \( g \in H^2 \) such that \( T_{\phi^{-1}} g = e_0 \). Then there exist \( u, v \in (H^2)^\perp \) such that \( \phi f = e_0 + u \) and \( \phi^{-1} g = e_0 + v \). By multiplication we have \( fg = e_0 + u + v + uv \), whence \( u + v + uv \in H^1 \) by Proposition 2. Since \( u, v \in (H^2)^\perp \), an easy calculation using Proposition 1 shows that \( u + v + uv = 0 \) and hence that \( fg = e_0 \).

**Widom’s Theorem.** Suppose \( \phi \in L^\infty \); then \( \sigma(T_\phi) \) is connected.

**Proof:** Consider the function \( f : C \setminus \sigma(T_\phi) \to H^2 \) given by the equations \( f(\lambda) = (\lambda - T_\phi)^{-1} e_0 \). Then \( f \) is differentiable and we have \( P[(\lambda - \phi)f'(\lambda) + f(\lambda)] = 0 \). But Proposition 6 gives also the equation \( P[1/((\lambda - \phi)f(\lambda))] = e_0 \). Multiplying, we get the differential equation
\[
f'(\lambda) = f(\lambda) P \left( \frac{1}{\phi - \lambda} \right).
\]

Note that any non-zero solution of this equation is a multiple of \( f \) by a non-zero function independent of \( \lambda \). So, using the F. & M. Riesz Theorem again, we solve to get, for any fixed \( \alpha \) in each connected component of \( C \setminus \sigma(T_\phi) \) and for each \( \lambda \) in that component,
\[
f(\lambda) = f(\alpha) \exp \left( P \int_{\Gamma} \frac{d\mu}{\phi - \mu} \right).
\]
where $\Gamma$ is any simple integration arc in the component going from $\alpha$ to $\lambda$. If $\Gamma$ is closed, the condition
\[ \exp \left( p \int_{\Gamma} \frac{d\mu}{\phi - \mu} \right) = e_0 \]
of Proposition 5 holds, so no such $\Gamma$ separates $\sigma(\phi)$ and connectedness of $\sigma(T_\phi)$ will follow if we can show that $\sigma(T_\phi)$ is exterior to such a $\Gamma$ whenever $\sigma(\phi)$ is. Suppose, then, that $\Gamma$ is a simple closed integration path in $\mathbb{C} \setminus \sigma(T_\phi)$ and that $\sigma(\phi)$ is exterior to $\Gamma$. Then the solution to the differential equation gives a unique analytic continuation of $f$ to the interior of $\Gamma$, so that, setting $Q$ to be the associated spectral idempotent for $T_\phi$, we have
\[ Qe_0 = \frac{1}{2\pi i} \int_{\Gamma} (\mu - T_\phi)^{-1} e_0 d\mu = \frac{1}{2\pi i} \int_{\Gamma} f(\mu) d\mu = 0. \]
Now $(\lambda - T_\phi)(e_n f(\lambda)) = e_n + \sum_{i=0}^{n-1} \beta_i e_i$ for each $\lambda \in \Gamma$ and some related scalars $\beta_i$; assuming inductively that $Qe_i = 0$ for $i < n$, it follows, since $Q$ commutes with $T_\phi$, that
\[ Q(e_n f(\lambda)) = (\lambda - T_\phi)^{-1} Qe_n, \]
and integration around $\Gamma$ gives $Qe_n = 0$. So $Q = 0$ by induction, whence no part of $\sigma(T_\phi)$ is interior to $\Gamma$ and the theorem is proved.

References


Micheal Ó Searcoid
Department of Mathematics
University College
Dublin
email: micheal.osearcoid@ucd.ie