# THE BAER-SPECKER GROUP

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The **Baer-Specker group**, **P**, is the group of functions from the natural numbers **N** into the integers **Z**. While **P** is very easy to define, it is the source of a wealth of problems, some of which have only recently been solved. The aim of this paper is to present an introductory account of old and new results about **P**, and to explore some of the connections of this group with other areas of mathematics, in particular with the real numbers, infinitary logic, and combinatorial set theory.

## Introduction

The Baer-Specker group **P** is an infinite abelian group under the addition

$$(f+g)(n) = f(n) + g(n)$$

for $n \in \mathbf{N}$. **P** contains the subgroup **S**, the direct sum of countably many copies of **Z**. In the literature, **P** is also denoted $\mathbf{Z}^{\mathbf{N}}$, or $\mathbf{Z}^{\omega}$. Since all the groups considered in this paper are abelian, it will save space to adopt the convention that the term group is short for abelian group. The textbooks [20] and [22] are good references for infinite abelian group theory. We use the symbols $\omega$, $\omega_1$, and $2^{\omega}$ to denote the cardinal numbers of the natural numbers **N**, the first uncountable cardinal, and the cardinal number of the real numbers **R**, respectively. For example, **P** has cardinality $2^{\omega}$, and **S** has cardinality $\omega$.

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There are four sections in the paper. The first resumes some basic material on **P**; the second looks at a recently discovered connection relating slender subgroups of **P** to certain cardinal invariants of the real numbers. In the third section, some logical aspects of **P** are explored. The final section is about the complexity of the lattice of subgroups of **P**. Since the material covered in the paper is diverse and scattered across different domains, I have not given many proofs, hoping that the bibliography will enable the reader to follow themes in more depth.

## 1. Basics

The structure of infinite free groups is relatively clear (see, for example, [22]): for each infinite cardinal $\kappa$, there is exactly one free group of cardinality $\kappa$ on $\kappa$ generators (up to isomorphism). So an obvious initial question in the study of **P** is to determine how it stands in relation to freeness: is **P** free?

**Definition** A group $G$ is **free** if $G$ is (isomorphic to) a direct sum of copies of **Z**.

For example, **P** has a free subgroup **S**.

**Theorem 1.1** (Baer [1]) *The group* **P** *is not free.*

We shall deduce this theorem from a stronger assertion below involving the notion of $\kappa$–freeness.

**Definition** Suppose that $\kappa$ is an infinite cardinal. A group $G$ is $\kappa$–**free** if every subgroup $H \leq G$ having less than $\kappa$ elements is free.

Every free group is $\kappa$–free for every cardinal $\kappa$, since subgroups of free groups are free; if $\lambda < \kappa$, then $\kappa$–freeness implies $\lambda$–freeness. Questions about whether $\lambda$–freeness implies $\kappa$–freeness for $\lambda < \kappa$ are highly non-trivial and have stimulated one of the most important research orientations in infinite abelian group theory, leading for example to Shelah's singular compactness theorem [7, 25, 20] and independence results in set theory [30]. Since in general $\kappa$–freeness is weaker than freeness, we can refine the initial question about the non-freeness of **P** and ask whether there are any cardinals $\kappa$ for which **P** is $\kappa$–free. Recall that $\omega_2$ is the second uncountable cardinal, the cardinal successor of $\omega_1$.

**Proposition 1.2** *The group* $\mathbf{P}$ *is not* $\omega_2$*–free.*
*Proof:* We need to find a non-free subgroup of $\mathbf{P}$ which has cardinality $\omega_1$. Let $p$ be a fixed prime number, and take a pure subgroup $H$ of cardinality $\omega_1$ containing $\mathbf{S}$ and such that every element $(n_1, n_2, \ldots)$ of $H$ has the property that the tail is divisible by arbitrarily high powers of $p$: $(\forall m)(\exists r)(\forall k > r)(p^m | n_k)$.[2] Then the quotient group $H/pH$ is a vector space over $\mathbf{F}_p$, the finite field of $p$ elements. It follows that $H$ is not free, for if $H$ were free, then $H/pH$ must have dimension (and hence cardinality) $\omega_1$; but every coset of $H/pH$ contains an element of $\mathbf{S}$, and hence $H/pH$ has cardinality at most $|\mathbf{S}| = \omega$ –a contradiction. So $H$ is not free. ■

We can use Proposition 1.2 to improve exercise 19.7 of [22]:

**Corollary** *For every uncountable cardinal* $\kappa$*, there exists a non-free* $\omega_1$*–free group of cardinality* $\kappa$*.*
*Proof:* For example, the group $\oplus_{\alpha<\kappa} H$, the direct sum of $\kappa$ copies of the group $H$ in the proof of Proposition 1.2, will work. ■

Proposition 1.2 implies Baer's theorem, since $H$ is a non-free subgroup of $\mathbf{P}$. However, it leaves open the question whether countable subgroups of $\mathbf{P}$ are free, i.e. whether $\mathbf{P}$ is $\omega_1$–free.

**Theorem 1.3** (Specker [42]) *The group* $\mathbf{P}$ *is* $\omega_1$*–free.*
*Proof:* This is a well-known result and a full proof is given in the standard reference textbooks [20, 22]. It rests on Pontryagin's Criterion: a countable group is free if and only if every finite rank subgroup is free. We shall give a proof of this criterion using logic in section three. The proof of Theorem 1.3 proceeds as follows: every finite rank subgroup of $\mathbf{P}$ is embedded in a finitely generated torsion-free direct summand of $\mathbf{P}$, and hence is free; so if $G \leq \mathbf{P}$ is countable, then every finite rank subgroup of $G$ is free; now apply Pontryagin's Criterion. ■

To close this section, let us introduce another type of "local" freeness which has been intensively studied: a group $G$ is almost free if $G$ is $|G|$–free ($|G|$ is the number of elements of $G$), i.e. every

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[2] Or: let $H$ be an elementary submodel of the $p$-adic closure of $\mathbf{S}$ in $\mathbf{P}$ of cardinality $\omega_1$ containing $\mathbf{S}$. $H$ exists by the Downward Loewenheim-Skolem Theorem of first-order logic.

subgroup of $G$ of smaller cardinality than $G$ is free. Is $\mathbf{P}$ almost free?

**Corollary 1.4** $\mathbf{P}$ *is almost free if and only if the Continuum Hypothesis* ($\mathbf{CH}$: $2^\omega = \omega_1$) *holds.*

If the Continuum Hypothesis is true, then $\mathbf{P}$ is almost free if $\mathbf{P}$ is $\omega_1$–free, which is true by Theorem 1.3; if the Continuum Hypothesis is false, then the subgroup $H$ in Proposition 1.2 is a non-free subgroup of cardinality $\omega_1 < 2^\omega = |\mathbf{P}|$ and so $\mathbf{P}$ is not almost free. Thus one cannot decide whether $\mathbf{P}$ is almost free or not on the basis of ordinary set theory (ZFC).

One trend in the study of $\kappa$–freeness has been to try to find equivalences between the algebraic and set-theoretic definitions. In light of Corollary 1.4, it might be interesting to know whether there are algebraic properties $\varphi$ and $\psi$ such that:

(1) $\mathbf{P}$ has almost $\varphi$ iff the weak Continuum Hypothesis ($\mathbf{wCH}$: $2^\omega < 2^{\omega_1}$) holds;

(2) $\mathbf{P}$ has almost $\psi$ iff Diamond holds.

Diamond $\diamond$ is a stronger form of the Continuum Hypothesis $\mathbf{CH}$. One way to state $\mathbf{CH}$ is as a list of guesses $A_\alpha$ for the subsets of $\mathbf{N}$:

$$(\exists\{A_\alpha \subseteq \alpha : \alpha < \omega_1\} \text{ such that}$$
$$(\forall X \subseteq \mathbf{N})(\{\alpha : \ X = A_\alpha\} \text{ is a stationary subset of } \omega_1)).$$

A stationary subset of $\omega_1$ is large: it intersects every closed unbounded subset of $\omega_1$ (in the order topology) non-trivially. In other words, the Continuum Hypothesis says that there is a list of length $\omega_1$ which predicts correctly every subset of natural numbers a large number of times. What about subsets of $\omega_1$? One cannot hope for a list of length $\omega_1$ which would predict correctly every subset of $\omega_1$, since there are $2^{\omega_1}$ subsets of $\omega_1$ and $2^{\omega_1} > \omega_1$. But perhaps one might be able to predict correctly just the initial segments of subsets of $\omega_1$. Diamond asserts that there is a list of $\omega_1$ guesses $A_\alpha$ for the initial segments of subsets of $\omega_1$ and these guesses are correct on a large subset of $\omega_1$. Formally, Diamond states:

$$(\exists\{A_\alpha \subseteq \alpha : \alpha < \omega_1\} \text{ such that}$$
$$(\forall X \subseteq \omega_1)(\{\alpha : \ X \cap \alpha = A_\alpha\} \text{ is a stationary subset of } \omega_1)).$$

A more detailed explanation of this type of prediction principle can be found in the standard textbooks on set theory [26] and [29].

## 2. P and the real numbers R

We start this section by looking at some cardinal invariants of the real numbers **R**. Recent research has uncovered rather surprising connections between these invariants and the size of certain subgroups of **P**.

**Definition** (1) The **additivity of measure**, **add(L)**, is the smallest number of measure-zero subsets of **R** whose union is not of measure zero.

(2) The **additivity of category**, **add(B)**, is the smallest number of first category subsets of **R** whose union is of second category (not of first category).

(3) The cardinal **d**, the **dominating number**, is defined as:

$$\min\{|D| : \ D \text{ is a subset of } \mathbf{N}^{\mathbf{N}} \text{ and}$$
$$(\forall g \in \mathbf{N}^{\mathbf{N}})(\exists f \in D)(g(n) \leq f(n) \text{ for all but finitely many } n)\}.$$

(4) The cardinal **b**, the **bounding number**, is defined as:

$$\min\{|B| : \ B \text{ is a subset of } \mathbf{N}^{\mathbf{N}} \text{ and}$$
$$(\forall g \in \mathbf{N}^{\mathbf{N}})(\exists f \in B)(g(n) < f(n) \text{ for all but finitely many } n)\}.$$

The countable additivity of Lebesgue measure and the Baire category theorem imply that **add(L)** and **add(B)** are both at least $\omega_1$. And they are also at most $2^\omega$. It is immediate too that $\omega_1 \leq \mathbf{b} \leq \mathbf{d} \leq 2^\omega$.

The reason why the dominating and bounding numbers are called invariants of the reals is that the irrationals are homeomorphic to the topological space $\mathbf{N}^{\mathbf{N}}$ when $\mathbf{N}^{\mathbf{N}}$ is given the product topology and $\mathbf{N}$ has the discrete topology.

We shall need one other invariant which is called the pseudo-intersection number. A family $F$ of subsets of $\mathbf{N}$ has the **strong finite intersection property** (SFIP) if the intersection of every finite subfamily is an infinite set. For example, the family of co-finite subsets of $\mathbf{N}$ has the SFIP. A set $A$ is **almost contained** in a set $B$ if $A \setminus B$ is finite.

**Definition** The cardinal $\mathbf{p}$ is

$$\begin{aligned}\min\{|F| : \;& F \text{ is a family of subsets of } \mathbf{N} \text{ such that } F \text{ has} \\ & \text{the SFIP but there is no infinite set which is almost} \\ & \text{contained in every member of } F\}.\end{aligned}$$

It is an instructive exercise to show that $\omega_1 \leq \mathbf{p} \leq 2^\omega$.

These cardinals are related as in the following picture, part of the Cichon diagram (except for the cardinal $\mathbf{p}$). An arrow relation $\kappa \to \lambda$ means $\kappa \leq \lambda$:

$$\begin{tikzcd}
 & 2^\omega \\
 & \mathbf{d} \arrow[u] \\
 & \mathbf{b} \arrow[u] \\
\mathbf{add(L)} \arrow[r, "\text{Bartoszynski [2], Raisonnier and Stern [36]}"'] & \mathbf{add(B)} \arrow[u, "\text{Miller [32]}"'] \\
\omega_1 \arrow[u] \arrow[r] & \mathbf{p} \arrow[u, "\text{Martin and Solovay [31]}"']
\end{tikzcd}$$

Some of the earliest results on the cardinal invariants of the reals are due to Rothberger [37]. The discovery of Martin's Axiom in 1970 stimulated renewed interest in these and other invariants. A fuller account of the area which has been studied in great depth by mathematicians since the 1970's is available in the articles by van Douwen [13] and Vaughan [44]. More recent work has revealed links between cardinal invariants and quadratic forms: several of these invariants determine how large orthogonal complements there are in a quadratic space. This research (on Gross and strongly Gross spaces) is surveyed in the paper [43]. The bounding number $\mathbf{b}$ also appears in recent work (in functional analysis) on metrizable barrelled spaces [38].

To explain the connection of cardinal invariants with the Baer-Specker group $\mathbf{P}$, we shall introduce one further definition.

**Definition** (Łoš). A group $G$ is **slender** if whenever $\phi$ is a homomorphism from $\mathbf{P}$ into $G$, then $\phi(e_n) = 0$ for all but finitely many $n$, where $e_n$ is the element of $\mathbf{P}$ that has 1 at the $n$–th co-ordinate and 0 everywhere else.

There are several important equivalent characterizations of slender groups which we insert here for the sake of completeness and as an aid to intuition.

**Theorem 2.1** (Nunke [33], Heinlein [24], Eda (1982, see [20])) The group $G$ is slender if and only if $G$ does not contain a copy of the rationals $\mathbf{Q}$, the cyclic group of order $p$ $\mathbf{Z}(\mathbf{p})$, the $p$-adic integers $\mathbf{J}_p$, or $\mathbf{P}$;
equivalently, every homomorphism $\phi$ from $\mathbf{P}$ into $G$ is continuous, where $G$ and $\mathbf{Z}$ are given the discrete topology and $\mathbf{P}$ the product topology;
equivalently, for any family $\{G_i : i \in I\}$ and homomorphism $\phi$ from $\Pi_{i\in I}G_i$ to $G$, there are $\omega_1$-complete ultrafilters $D_1, \ldots, D_n$ on $I$ such that

$$(\forall g \in \Pi_{i\in I}G_i)(\text{if the support of } g, \{i \in I : g(i) \neq 0\},$$
$$\text{does not belong to } D_k (1 \leq k \leq n), \text{ then } \phi(g) = 0).$$

**Corollary 2.2** *Every $\omega_1$–free group which does not contain* $\mathbf{P}$ *is slender.*

**Examples** The group $\mathbf{Z}$ is slender; every free group is slender. $\mathbf{P}$ is not slender (Specker [42]). Subgroups and direct sums of slender groups are slender.

Specker's proof that $\mathbf{P}$ is not slender works for many other subgroups of $\mathbf{P}$ which exhibit the **Specker phenomenon**. But these subgroups all have cardinality $2^\omega$.

**Definition** (Eda [15], Blass [8]) (1) A subgroup $G$ of $\mathbf{P}$ exhibits the **Specker phenomenon** iff $G$ contains a sequence $\{g_n : n \in \mathbf{N}\}$ of linearly independent elements such that whenever $\phi$ is a homomorphism from $G$ into $\mathbf{Z}$, then $\phi(g_n) = 0$ for all but finitely many $n$.

(2) The **Specker-Eda number, se**, is defined as:

$$\min\{|G| : G \leq \mathbf{P} \text{ exhibits the Specker phenomenon}\}.$$

**Corollary 2.3** $\omega_1 \leq \mathbf{se} \leq 2^\omega$.

**Theorem 2.4** (Eda [15].) (1) **CH** *implies* $\mathbf{se} = \omega_1$.

(2) *Martin's Axiom* (**MA**) *implies* $\mathbf{se} = 2^\omega$.

(3) *There is a model of ordinary set theory* (ZFC) *in which* $\mathbf{se} < 2^\omega$.

In 1994, Andreas Blass observed that Eda's proofs establish connections between the Specker-Eda number and some of the cardinal invariants of the real numbers which were defined at the beginning of this section.

**Theorem 2.5** (Eda [15], Blass [8]) (1) $\mathbf{p} \leq \mathbf{se} \leq \mathbf{d}$.

(2) $\mathbf{add}(\mathbf{L}) \leq \mathbf{se} \leq \mathbf{b}$.

**Conjecture 2.6** (Blass [8]) $\mathbf{se} = \mathbf{add}(\mathbf{B})$.

### 3. P and infinitary logic

From the logical point of view, a group $G$ is a structure $\mathbf{G} = (G, +^G, -^G, 0^G)$ which satisfies the axioms for a group. In this section we use $\mathbf{L}$ to denote the vocabulary of groups, that is, $\mathbf{L}$ contains the constant, unary, and binary function symbols $\mathbf{0}$, $-$, and $+$ to name the zero, inverse, and addition of a group. There are other possible choices for the vocabulary of groups, but for the sake of definiteness we shall fix $\mathbf{L}$ as above. It is also a fact that all the theorems of logic presented here are true in much greater generality.

The infinitary language $\mathbf{L}_{\infty\kappa}$ is the smallest class of formulas in the vocabulary $\mathbf{L}$ which is closed under negations, conjunctions of arbitrary length, and strings of quantifiers

$$(\exists x_1 \exists x_2 \ldots \exists x_\alpha \ldots)_{(\alpha<\lambda)}$$

of length $\lambda$ less than $\kappa$. This infinitary language is more expressive than the first-order language of groups where one is limited to negations, finite conjunctions, and finite strings of quantifiers. While the axioms for a group are first-order, many of the interesting group-theoretic properties cannot be expressed by first-order sentences. For example, the following sentence of $\mathbf{L}_{\infty\omega}$ says that a group is torsion:

$$(\forall g)(g = 0 \text{ or } 2g = 0 \text{ or } 3g = 0 \text{ or } \ \ldots \ \text{ or } ng = 0 \ldots).$$

But the concept of torsion cannot be axiomatized in a first-order language. Infinitary logic can express the concepts of $\kappa$–freeness, $\kappa$–purity, $< \kappa$–generatedness and so on. The paper by Barwise [3] is an excellent introduction to the back and forth methods characteristic of infinitary logic. Other useful references for infinitary logics are the book by Barwise [4] and the article by Dickmann [12]. A typical problem in general infinitary model theory involves determining whether there are infinitarily equivalent nonisomorphic models in various cardinalities, [40].

**Definition** Two groups $A$ and $B$ are $\mathbf{L}_{\infty\kappa}$**–equivalent** if and only if for every sentence $\varphi$ in $\mathbf{L}_{\infty\kappa}$: $\varphi$ is true in $A$ iff $\varphi$ is true in $B$.

So $\mathbf{L}_{\infty\kappa}$–equivalent groups cannot be distinguished by a sentence in the infinitary language $\mathbf{L}_{\infty\kappa}$. There is an algebraic characterization of infinitary equivalence which is very useful and perhaps more familiar.

**Theorem 3.1** (Karp [27], Benda [6], Calais [9]) *Two groups $A$ and $B$ are $\mathbf{L}_{\infty\kappa}$–equivalent if and only if there is a $\kappa$–***extendible system of partial isomorphisms** *from $A$ to $B$.*

A $\kappa$–extendible system of partial isomorphisms from $A$ to $B$ is a family $F$ of isomorphisms between subgroups of $A$ and $B$ which has the $\kappa$–back-and-forth property: if $\phi \in F$ is an isomorphism from $M_1 \leq A$ onto $N_1 \leq B$ and $X$ (respectively $Y$) is a subset of $A$ (respectively $B$) of cardinality less than $\kappa$, then there exists $\psi \in F$ from $M_2$ onto $N_2$ such that $M_1 \leq M_2 \leq A$, $N_1 \leq N_2 \leq B$, $\psi$ extends $\phi$ and $X$ $(Y)$ is a subset of $M_2$ $(N_2)$.

Using this algebraic concept, it is easy to check for example that any two uncountable free groups are $\mathbf{L}_{\infty\omega}$–equivalent. $\kappa$–extendible systems are a natural generalization of Cantor's technique for showing that there is (up to isomorphism) exactly one unbounded dense countable linear order, namely the linearly ordered set of the rational numbers [3]. Indeed, for countable groups, infinitary equivalence and isomorphism are synonymous:

**Theorem 3.2** (Scott [39]) *If $A$ and $B$ are countable $\mathbf{L}_{\infty\omega}$–equivalent groups, then $A$ and $B$ are isomorphic.*

The infinitary model theory of abelian groups was intensively studied in the 1970's by Barwise, Eklof, Fischer, Gregory, Kueker, and Mekler (see [5, 16, 17, 23, 28] for example). One of the first important results is due to Eklof, who succeeded in determining which groups are infinitarily equivalent to free groups.

**Definition** A subgroup $A$ of $G$ is $\kappa$–pure if for every subgroup $B$ such that $A \leq B \leq G$ and $B/A$ is $< \kappa$–generated (i.e. generated by fewer than $\kappa$ elements), $A$ is a direct summand of $B$.

**Theorem 3.3** (Eklof [16]) *A group $G$ is $\mathbf{L}_{\infty\kappa}$–equivalent to a free group if and only if every $< \kappa$–generated subgroup of $G$ is contained in a free, $\kappa$–pure subgroup of $G$.*

For the purposes of this exposition, it is sufficient to know the following corollaries.

**Corollary 3.4** *A group $G$ is $\mathbf{L}_{\infty\omega}$–equivalent to a free group iff every subgroup of $G$ of finite rank is free.*

Eklof used this result to deduce a very famous criterion for freeness in countable groups:

**Corollary 3.5** (Pontryagin's Criterion [35]) *A countable group is free iff every subgroup of finite rank is free.*

*Proof:* Apply Scott's Theorem 3.2 to Corollary 3.4. ■

**Corollary 3.6** (Kueker) *A group is $\mathbf{L}_{\infty\omega}$–equivalent to a free group iff it is $\omega_1$–free.*

Now we can return to the Baer-Specker group **P** and see what these facts tell us.

**Corollary 3.7** (Keisler-Kueker) *The Baer-Specker group* **P** *is $\mathbf{L}_{\infty\omega}$–equivalent to a free group. The class of free groups is not definable in $\mathbf{L}_{\infty\omega}$.*

**Corollary 3.8** (Eklof [16]) *The group* **P** *is not $\mathbf{L}_{\infty\omega_1}$–equivalent to a free group.*

Since free groups are slender, it follows too from Corollary 3.7 that the class of slender groups is not definable in $\mathbf{L}_{\infty\omega}$. It might be tempting to conjecture that **P** is not $\mathbf{L}_{\infty\omega_1}$–equivalent to a slender group. Another possible suggestion is that **P** is not $\mathbf{L}_{\infty\mathbf{se}}$–equivalent to a slender group. Mekler showed that if $\kappa$ is a strongly compact cardinal, then the class of free groups is definable in $\mathbf{L}_{\infty\kappa}$. This prompts the question whether the class of slender groups is definable in $\mathbf{L}_{\infty\kappa}$ if $\kappa$ is strongly compact. Eklof and Mekler have developed applications of other generalized logics to problems of abelian group theory in the papers [18] and [19].

## 4. The lattice of subgroups of P

The broad thrust in this section is to describe some recent research on the complexity of the lattice of subgroups of **P**. One way to measure this complexity is to study what sorts of groups can be embedded into **P**. Very generally, the natural questions often have the form whether there are families of maximal possible size of subgroups of **P** which are strongly different (non-isomorphic) in some precisely defined sense.

An example of this type of theorem in the context of general abelian groups is the following.

**Theorem 4.1** (Eklof, Mekler and Shelah [21]) *Under various set-theoretic hypotheses, there exist families of maximal possible size of almost free abelian groups which are pairwise almost disjoint (the intersection of any pair contains no non-free subgroup).*

There is an inverse correlation between the size of the family and the strong difference of its members. If one considers families of pure subgroups of $\mathbf{P}$ and takes the notion of strong difference to mean that the only homomorphisms between any pair are those of finite rank, then it is possible to prove the existence of a strongly different family of maximal size.

**Theorem 4.2** (Corner and Goldsmith [10]) *Let $\mathbf{D}$ be the subgroup of $\mathbf{P}$ containing $\mathbf{S}$ such that $\mathbf{D}/\mathbf{S}$ is the divisible part of $\mathbf{P}/\mathbf{S}$. Let $c = 2^\omega$. There exists a family $\mathbf{C}$ consisting of $2^c$ pure subgroups $G$ of $\mathbf{D}$ with $\mathbf{D}/G$ rank–1 divisible where each $G$ is slender, essentially-indecomposable, essentially-rigid with* $\mathrm{End}(G) = \mathbf{Z} + E_0(G)$, *where $E_0(G)$ is the ideal of all endomorphisms of $G$ whose images have finite rank.*

A similar type of question is the following: does there exist a family of $2^{\omega_1}$ non-isomorphic pure subgroups of $\mathbf{P}$, each of cardinality $\omega_1$, such that the intersection of any pair is free? The answer is positive.

**Theorem 4.3** (Shelah and Kolman [41]) *There exists a family $\{G_\alpha : \alpha < 2^{\omega_1}\}$ of pure subgroups of $\mathbf{P}$ such that*
(1) *each $G_\alpha$ has cardinality $\omega_1$;*
(2) *if $\alpha \neq \beta$, then $G_\alpha \cap G_\beta$ is free.*

The question whether certain classes of group can be embedded in $\mathbf{P}$ sometimes leads to independence results. Recall that a group $G$ is $\omega_1$–separable if every countable subset of $G$ is contained in a free direct summand of $G$.

**Theorem 4.4** (Dugas and Irwin [14]) *Embeddability of $\omega_1$–separable groups of cardinality $\omega_1$ in $\mathbf{P}$ is independent of ZFC.*

Reflexive subgroups of $\mathbf{P}$ have also been studied in some depth. The following result was known for many years under the additional assumption of the Continuum Hypothesis.

**Theorem 4.5** (Ohta [34]) *There exists a non-reflexive dual subgroup of $\mathbf{P}$.*

The variety displayed in this selection of results on the Baer-Specker group $\mathbf{P}$ illustrates how this easily definable abelian group is a source of interesting research problems whose solutions often reveal unexpected connections with problems in other domains of mathematics.

**References**


[1] R. Baer, *Abelian groups without elements of finite order*, Duke Math. J. **3** (1937), 68-122.

[2] T. Bartoszynski, *Additivity of measure implies additivity of category*, Trans. Amer. Math. Soc. **281** (1984), 209-213.

[3] J. Barwise, *Back and forth through infinitary logic*, pp.5-33 *in*: M. Morley (ed.), Studies in Model Theory, 1975.

[4] J. Barwise, Admissible Sets and Structures. Springer-Verlag: Berlin, 1975.

[5] J. Barwise and P. C. Eklof, *Infinitary properties of abelian torsion groups*, Ann. Math. Logic **2** (1970), 25-68.

[6] M. Benda, *Reduced products and non-standard logics*, J. Sym. Logic **34** (1969), 424-436.

[7] S. Ben-David, *On Shelah's compactness of cardinals*, Israel J. Math. **31** (1978), 34-56.

[8] A. Blass, *Cardinal characteristics and the product of countably many infinite cyclic groups*, J. Algebra **169** (1994), 512-540.

[9] J. P. Calais, *La méthode de Fraïssé dans les langages infinis*, C. R. Acad. Sci. Paris **268** (1969), 785-788.

[10] A. L. S. Corner and B. Goldsmith, *On endomorphisms and automorphisms of some pure subgroups of the Baer-Specker group*, pp.69-78 *in*: R. Goebel *et al.* (eds.), Abelian group theory and related topics, Contemp. Math. **171** (1994).

[11] K. J. Devlin, Constructibility. Springer-Verlag: Berlin, 1984.

[12] M. Dickmann, *Larger infinitary languages*, Chapter IX *in*: J. Barwise and S. Feferman (eds.), Model-theoretic Logics. Springer-Verlag: Berlin, 1985.

[13] E. van Douwen, *The integers and topology*, pp.111-167 *in*: K. Kunen and J. E. Vaughan (eds.), Handbook of Set-theoretic Topology. North-Holland: Amsterdam, 1984.

[14] M. Dugas and J. Irwin, *On pure subgroups of cartesian products of integers*, Results in Math. **15** (1989), 35-52.

[15] K. Eda, *A note on subgroups of* $\mathbf{Z}^{\mathbf{N}}$, pp.371-374 *in*: R.Goebel *et al.* (eds.), Abelian Group Theory. Lecture Notes in Mathematics 1006. Springer-Verlag: Berlin, 1983.

[16] P. C. Eklof, *Infinitary equivalence of abelian groups*, Fund. Math. **81** (1974), 305-314.

[17] P. C. Eklof and E. R. Fisher, *The elementary theory of abelian groups*, Ann. Math. Logic **4** (1972), 115-171.

[18] P. C. Eklof and A. H. Mekler, *Infinitary stationary logic and abelian groups*, Fund. Math. **112** (1981), 1-15.

[19] P. C. Eklof and A. H. Mekler, *Categoricity results for* $L_{\infty\kappa}$*–free algebras*, Ann. Pure and Appl. Logic **37** (1988), 81-99.

[20] P. C. Eklof and A. H. Mekler, Almost Free Modules. North-Holland: Amsterdam, 1990.

[21] P. C. Eklof, A. H. Mekler and S. Shelah, *Almost disjoint abelian groups*, Israel J. Math. **49** (1984), 34-54.

[22] L. Fuchs, Infinite Abelian Groups, Vol. I. Academic Press: New York, 1970.

[23] J. Gregory, *Abelian groups infinitarily equivalent to free groups*, Notices Amer. Math. Soc **20** (1973), A-500.

[24] G. Heinlein, Vollreflexive Ringe und schlanke Moduln, doctoral dissertation, Univ. Erlangen, 1971.

[25] W. Hodges, *In singular cardinality, locally free algebras are free*, Algebra Universalis **12** (1981), 205-259.

[26] T. Jech, Set Theory. Academic Press: New York, 1978.

[27] C. Karp, *Finite quantifier equivalence*, pp.407-412 *in*: J. Addison *et al.* (eds.), The Theory of Models. North-Holland: Amsterdam, 1965.

[28] D. W. Kueker, $L_{\infty\omega_1}$–elementarily equivalent models of power $\omega_1$, pp. 120-131 *in*: Logic Year 1979-80, Lecture Notes in Mathematics 859. Springer-Verlag: Berlin, 1981.

[29] K. Kunen, Set Theory. North-Holland: Amsterdam, 1980.

[30] M. Magidor and S. Shelah, *When does almost free imply free?*, J. Amer. Math. Soc. **7** (1994), 769-830.

[31] D. A. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic **2** (1970), 143-178.

[32] A. W. Miller, *Additivity of measure implies dominating reals*, Proc. Amer. Math. Soc. **91** (1984), 111-117.

[33] R. Nunke, *Slender groups*, Acta Sci. Math. (Szeged) **23** (1962), 67-73.

[34] H. Ohta, *Chains of strongly non-reflexive dual groups of integer-valued continuous functions*, Proc. Amer. Math. Soc. **124** (1996), 961-967.

[35] L. S. Pontryagin, *The theory of topological commutative groups*, Ann. Math. **35** (1934), 361-388.

[36] J. Raisonnier and J. Stern, *The strength of measurability hypotheses*, Israel J. Math. **50** (1985), 337-349.

[37] F. Rothberger, *Eine Aequivalenz zwischen der Kontinuumhypothese und der Existenz der Lusinschen und Sierpinskischen Mengen*, Fund. Math. **30** (1938), 215-217.

[38] S. A. Saxon and L. M. Sánchez-Ruiz, *Barrelled countable enlargements and the bounding cardinal*, J. London Math. Soc. **53** (1996), 158-166.

[39] D. Scott, *Logic with denumerably long formulas and finite strings of quantifiers*, pp.329-341 *in*: J. Addison *et al.* (eds.), The Theory of Models. North-Holland: Amsterdam, 1965.

[40] S. Shelah, *Existence of many $L_{\infty\lambda}$–equivalent, non-isomorphic models of $T$ of power $\lambda$*, Ann. Pure and Appl. Logic **34** (1987), 291-310.

[41] S. Shelah and O. Kolman, *Almost disjoint pure subgroups of the Baer-Specker group*, submitted.

[42] E. Specker, *Additive Gruppen von Folgen ganzer Zahlen*, Portugaliae Math. **9** (1950), 131-140.

[43] O. Spinas, *Cardinal invariants and quadratic forms*, pp.563-581 *in*: H. Judah (ed.), Set Theory of the Reals. Israel Mathematical Conference Proceedings (Gelbart Research Institute, Bar-Ilan University), 1993.

[44] J. E. Vaughan, *Small uncountable cardinals and topology*, pp.195-218 *in*: J. van Mill and G. M. Reed (eds.), Open Problems in Topology. North-Holland: Amsterdam, 1990.

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