# DESCRIBING IDEALS OF ENDOMORPHISM RINGS

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## 1. Introduction

It is well known that the ring of linear transformations of a finite dimensional vector space is simple, i.e. it has no non–trivial proper two–sided ideals. It is, perhaps, not so well known that the (two–sided) ideals in the ring of linear transformations of an infinite dimensional vector space can be characterized by a single cardinal invariant, [1]. It is therefore reasonably natural to ask if there is a generalization of Baer's result to ideals in the endomorphism ring of a wider class of modules. The purpose of this present work is to explore this possibility.

A first generalization is to replace the underlying field of the vector space by a ring. A natural extension of the concept of a field is a discrete valuation ring since a discrete valuation ring modulo its Jacobson radical is a field. Recall (see e.g. [5]) that $R$ is a discrete valuation ring if $R$ is a principal ideal ring with exactly one maximal ideal or, alternatively, with one prime element $p$. In particular the ring of $p$–adic rationals, $\mathbf{Z}_p$, is a discrete valuation ring and the reader may replace all references to discrete valuation rings with the $p$–adic rationals without any serious loss in generality. Also, recall that if $E$ is a ring then the Jacobson radical of $E$ is defined to be the intersection of all maximal ideals of $E$. It is well known that this is equivalent to the set of all elements $x \in E$ such that $rxs$ is quasi–regular for all $r$ and $s$ in $E$, i.e. those elements $x$ for which $1 - rxs$ is a unit.

Since a vector space over a field $F$ is a free $F$–module, an obvious question to ask is whether a corresponding characterization of ideals in the endomorphism ring of a free module over a discrete valuation ring exists. We address this problem in §3 and obtain Baer's Theorem for vector spaces as a corollary to our main result Theorem 3.3.

Whilst free modules over a discrete valuation ring are an obvious generalization of vector spaces it is, perhaps, not so obvious that complete modules over a complete discrete valuation ring $R$ have many similar properties to vector spaces (cf. [4]). Recall that a complete discrete valuation ring is a discrete valuation ring which is complete in its $p$–adic topology (i.e. the linear topology with basis of neighbourhoods of 0 given by $p^nR$, $n \geq 0$), where $p$ is the only prime element of $R$. In §4 we discuss the ideal structure of the endomorphism ring of such modules and achieve a complete characterization modulo the Jacobson radical.

We conclude this introduction by reviewing a number of standard concepts in module/abelian group theory. If $R$ is discrete valuation ring with prime element $p$ then we say that an $R$–submodule $H$ of the $R$–module $G$ is *pure* in $G$ if $H \cap p^nG = p^nH$ for all $n \geq 0$. If we consider the module $G$ as a topological module furnished with the $p$–adic topology (i.e. a basis of neighbourhoods of 0 is given by $p^nG$, $n \geq 0$), then $H$ is pure in $G$ precisely if the $p$–adic topology on $H$ coincides with the induced subspace topology. Also an $R$–module $D$ is *divisible* if given any $d \in D$ we can solve the equation $p^nx = d$ in $D$; a module is *reduced* if it has no non–trivial divisible submodules. (It is well known that a torsion–free divisible $R$–module is a direct sum of copies of the quotient field $Q$ of $R$.) Notice that if $H \subseteq G$ then $G/H$ divisible is equivalent to $H$ being dense in the $p$–adic topology on $G$.

Finally note that maps are written on the right and the word ideal will always mean a two–sided ideal.

## 2. Preliminaries

In this section we state a few well–known results on modules over complete discrete valuation rings. Throughout the section $R$ shall be a complete discrete valuation ring with prime element $p$. In considering torsion–free $R$–modules, the concept of a basic submodule is useful; a submodule $B$ of a torsion–free $R$–module $A$ is called a *basic submodule* of $A$ if $B$ is a free $R$–module such that

$A/B$ is divisible and $B$ is pure in $A$.

The first lemma is due to R. B. Warfield, [8]; it tells us how to obtain a basic submodule from the $R/pR$–vector space $A/pA$.

**Lemma 2.1** *Let $A$ be a torsion–free $R$–module and $\pi : A \longrightarrow A/pA$ the natural epimorphism. Let $\{x_i \mid i \in I\}$ be an $R/pR$–basis of $A/pA$ and choose $y_i \in A$ $(i \in I)$ such that $y_i\pi = x_i$. Then the submodule $B$ generated by $\{y_i \mid i \in I\}$ is a basic submodule of $A$. Moreover every basic submodule of $A$ arises in this way.* ∎

The lemma ensures the existence of a basic submodule $B$ and the uniqueness of its rank $\mathrm{rk}(B)$ where $\mathrm{rk}(B)$ is the usual rank of a free $R$–module. Hence we may define the *rank* of a torsion–free $R$–module $A$ as the rank of its basic submodule, i.e. $\mathrm{rk}(A) = \mathrm{rk}(B)$. Since the divisibility of the quotient module $A/B$ is equivalent to the density of $B$ in $A$ in the $p$–adic topology we obtain the following well–known characterization of complete reduced torsion–free $R$–modules; (2) is essentially due to I. Kaplansky, [5], and (3) is similar to a result proved for $p$–groups by H. Leptin, [6].

**Proposition 2.2** *The following properties of a reduced torsion–free $R$–module $A$ are equivalent:*
(1) *$A$ is complete.*
(2) *If $B$ is a basic submodule of $A$, then $A$ is the completion of $B$.*
(3) *If $B$ is a basic submodule of $A$, then every $R$–homomorphism $B \longrightarrow A$ extends uniquely to an $R$–endomorphism of $A$.* ∎

Next we state a few facts on complete reduced torsion–free $R$–modules; the proofs can be found in [2], [5], and [7].

**Lemma 2.3** *Let $A$ and $A'$ be complete reduced torsion–free $R$–modules.*
(1) *If $\theta$ is an endomorphism of $A$, then both $\ker\theta$ and $\mathrm{im}\,\theta$ are complete.*
(2) *If $S$ is a pure submodule of $A$ and is complete, then $S$ is a direct summand of $A$.*
(3) *$A$ is isomorphic to $A'$ if and only if $\mathrm{rk}(A) = \mathrm{rk}(A')$.* ∎

We finish this section with a standard result on the Jacobson radical of the endomorphism ring of a complete reduced torsion–free $R$–module; a proof may be found in [7].

**Proposition 2.4** *Let $E$ denote the endomorphism ring* $\mathrm{End}(A)$ *of the complete reduced torsion–free $R$–module $A$. Then*
(1) $J(E) = pE = Ep$;
(2) $E/J(E) \cong \mathrm{End}_{R/pR}(A/pA)$;
(3) $J(E) = \{\phi \in E \mid A\phi \subseteq pA\}$. ■

## 3. Free modules over discrete valuation rings

Here we shall discuss ideals of endomorphism rings of free $R$–modules over discrete valuation rings $R$. We will deduce Baer's Theorem on vector spaces as a corollary to our main result. Before we restrict our attention to modules over discrete valuation rings we prove a result (Proposition 3.2) which is true for modules in general. The definition of a direct endomorphism will be useful. An endomorphism $\mu$ of $A$ is called *k–direct* [*im–direct*] if $\ker(\mu)$ [$\mathrm{Im}(\mu)$] is a direct summand of $A$, and $\mu$ is said to be a *direct endomorphism* if it is both k–direct and im–direct. First we need:

**Lemma 3.1** *If $\sigma$ is a direct endomorphism of $A$, then there exists an endomorphism $\eta$ of $A$ such that*
(a) *$\sigma\eta$ and $\eta\sigma$ are both idempotent, and*
(b) $\mathrm{Im}(\sigma) = \mathrm{Im}(\eta\sigma)$, $\mathrm{Im}(\sigma\eta) = \mathrm{Im}(\eta)$, $\ker(\eta) = \ker(\eta\sigma)$, *and* $\ker(\sigma\eta) = \ker(\sigma)$.

*Proof:* By our assumption, we may write

$$A = \mathrm{Im}(\sigma) \oplus S \text{ and } A = \ker \sigma) \oplus T.$$

Then the restriction $\sigma_T : T \longrightarrow \mathrm{Im}(\sigma)$ is an isomorphism. Hence there exists $\tau : \mathrm{Im}(\sigma) \longrightarrow T$ such that

$$\tau(\sigma_T) = \mathrm{id}_{\mathrm{Im}(\sigma)} \text{ and } (\sigma_T)\tau = \mathrm{id}_T.$$

Now let $\eta : A \longrightarrow A$ be defined by

$$\eta_{\mathrm{Im}(\sigma)} = \tau \text{ and } \ker \eta = S.$$

We shall see that $\eta$ is the required endomorphism.

First we show that $\sigma\eta$ and $\eta\sigma$ are idempotent. Let $x$ be an arbitrary element of $A$. Then

$$x(\sigma\eta)^2 = x\sigma\eta\sigma\eta = x\sigma\tau\sigma\eta = x\sigma\eta,$$

since $\tau\sigma$ is the identity on $\mathrm{Im}(\sigma)$. So $(\sigma\eta)^2 = \sigma\eta$. Now let $x = a\sigma + s \in A$, where $a \in A$ and $s \in S$. Then

$$x(\eta\sigma)^2 = x\eta\sigma\eta\sigma = a\sigma\eta\sigma\eta\sigma = (a\sigma)\eta\sigma = x\eta\sigma.$$

Thus, $(\eta\sigma)^2 = \eta\sigma$, as required.

To prove part (b) we make the following calculations:

$$\begin{aligned}
\underline{\mathrm{Im}(\eta\sigma)} &= A\eta\sigma = A\sigma\eta\sigma = A\sigma\tau\sigma = A\sigma = \underline{\mathrm{Im}(\sigma)};\\
\underline{\mathrm{Im}(\sigma\eta)} &= A\sigma\eta = A\sigma\tau = T = \underline{\mathrm{Im}(\eta)};\\
\underline{\ker(\eta\sigma)} &= \{x \in A | x\eta\sigma = 0\} = \{x \in A | x\eta \in \ker(\sigma)\}\\
&= \{x \in A | x\eta \in \ker(\sigma) \cap T = 0\} = \underline{\ker(\sigma)};\\
\underline{\ker(\sigma\eta)} &= \{x \in A | x\sigma\eta = 0\} = \{x \in A | x\sigma\tau = 0\}\\
&= \{x \in A | x\sigma = 0\} = \underline{\ker(\sigma)}.
\end{aligned}$$

This completes the proof. ■

**Proposition 3.2** *Let $I$ be an ideal of the endomorphism ring* $\mathrm{End}(A)$ *of $A$ such that all $\mu \in I$ are k–direct. Moreover assume that $I$ contains a direct endomorphism $\sigma$. If $\alpha$ is an im–direct endomorphism of $A$ such that* $\mathrm{Im}(\alpha)$ *is isomorphic to a direct summand of* $\mathrm{Im}(\sigma)$*, then $\alpha$ belongs to the ideal $I$.*

*Proof*: Let $I$, $\sigma$, and $\alpha$ be as above. Then we can write

$$A = \mathrm{Im}(\sigma) \oplus S = \mathrm{Im}(\alpha) \oplus T \text{ and } \mathrm{Im}(\sigma) = R \oplus C,$$

where $R \cong_{\phi'} \mathrm{Im}(\alpha)$. We extend $\phi'$ to an endomorphism $\phi$ of $A$ by

$$\phi_R = \phi' \text{ and } \phi_{C\oplus S} = 0.$$

Now consider the endomorphism $\sigma\phi \in I$. Since $\sigma\phi$ belongs to $I$ it is k–direct. Moreover,

$$A\sigma\phi = (R \oplus C)\phi = R\phi = R\phi' = \mathrm{Im}(\alpha),$$

which is a direct summand of $A$. Hence $\sigma\phi$ is direct and we may apply Lemma 3.1 to $\sigma\phi$. Thus there exists $\eta \in \mathrm{End}(A)$ such that

$\eta\sigma\phi$ is an idempotent and $\mathrm{Im}(\eta\sigma\phi) = \mathrm{Im}(\sigma\phi) = \mathrm{Im}(\alpha)$. Therefore, for any $x \in A$, there is $y \in A$ with $x\alpha = y\eta\sigma\phi$. Hence,

$$(x\alpha)\eta\sigma\phi = y(\eta\sigma\phi)^2 = y\eta\sigma\phi = x\alpha,$$

which implies that $\alpha = \alpha\eta\sigma\phi$ is in $I$, since $I$ is an ideal. ■

Now let $R$ be a discrete valuation ring and $A$ a free $R$–module. The next theorem tells us something about ideals $I$ containing a direct endomorphism of a given rank. Recall that the *rank* of an endomorphism $\sigma$ is defined to be the rank of the free $R$–module $\mathrm{Im}(\sigma)$.

**Theorem 3.3** *Let $I$ be an ideal of* $\mathrm{End}(A)$. *If I contains a direct endomorphism of rank $\kappa$, then I contains all endomorphisms of rank less than or equal to $\kappa$.*

*Proof:* First we show that all endomorphisms of $A$ are k–direct. Let $\mu$ be any endomorphism of $A$. Then $A/\ker(\mu) \cong \mathrm{Im}(\mu)$, where $\mathrm{Im}(\mu)$ is free and hence projective. Thus there exists a homomorphism $\phi : \mathrm{Im}(\mu) \longrightarrow A$ with $\phi\mu = \mathrm{id}_A$. We show that $A = \ker(\mu) \oplus (\mathrm{Im}(\mu))\phi$. Let $x \in \ker(\mu) \cap (\mathrm{Im}(\mu))\phi$. Then there is $y \in A$ such that $x = y\mu\phi$ and

$$0 = x\mu = y\mu\phi\mu = y\mu,$$

since $\phi\mu = \mathrm{id}_A$, and thus $x = y\mu\phi = 0\phi = 0$. Also,

$$a = a\mu\phi + (a - a\mu\phi), \text{ with } (a - a\mu\phi)\mu = a\mu - a\mu\phi\mu = 0$$

for any $a \in A$, again since $\phi\mu = \mathrm{id}_A$. Thus $\ker(\mu)$ is a direct summand of $A$, implying that $\mu$ is k–direct.

Now let $\sigma \in I$ be a direct endomorphism of $A$ of rank $\kappa$, i.e. $A = \mathrm{Im}(\sigma) \oplus S$ and $\mathrm{rk}(\mathrm{Im}(\sigma)) = \kappa$. Next we prove that all direct (im–direct) endomorphisms $\alpha$ with $\mathrm{rk}(\alpha) \leq \kappa$ are elements of $I$. Let $\alpha$ be such an endomorphism. Then

$$A = \mathrm{Im}(\alpha) \oplus T \text{ and } \mathrm{rk}(\alpha) = \mathrm{rk}(\mathrm{Im}(\alpha)) \leq \kappa.$$

Since $\mathrm{Im}(\alpha)$ and $\mathrm{Im}(\sigma)$ are free $R$–modules with $\mathrm{rk}(\mathrm{Im}(\alpha)) \leq \mathrm{rk}(\mathrm{Im}(\sigma))$, there exists a direct summand of $\mathrm{Im}(\sigma)$ which is isomorphic to $\mathrm{Im}(\alpha)$. Thus we may apply Proposition 3.2, which implies that $\alpha \in I$.

Finally let $\phi$ be any endomorphism of $A$ with $\mathrm{rk}(\phi) \leq \kappa$. Since $\phi$ is k–direct, we may express $A$ as $A = \ker(\phi) \oplus C$, where $C \cong A/\ker(\phi) \cong \mathrm{Im}(\phi)$. Thus

$$\mathrm{rk}(C) = \mathrm{rk}(\phi) \leq \kappa.$$

Let $\pi$ be the projection of $A$ onto $C$ with $\ker(\pi) = \ker(\phi)$. Obviously $\pi$ is a direct endomorphism with $\mathrm{rk}(\pi) = \mathrm{rk}(C) \leq \kappa$. Thus $\pi \in I$. Hence $\phi = \pi\phi$ is an element of $I$ and this completes proof. ■

Note that the previous theorem holds for free modules over any ring $R$ having the property that submodules of free modules are free; e.g. all principal ideal domains have this property. So, in particular, Theorem 3.3 holds for a field. In this case we get even more, namely, we can characterize the ideals of the endomorphism ring $\mathrm{End}(A)$ of a vector space $A$.

**Corollary 3.4** *Let $A$ be a vector space over a field $R$. Then the only ideals of* $\mathrm{End}(A)$ *are the ideals $E_\kappa$ ($\kappa \geq \aleph_0$) defined by $E_\kappa = \{\alpha \in \mathrm{End}(A) |\, \mathrm{rk}(\alpha) < \kappa\}$.*

*Proof:* Note first that all endomorphisms of a vector space $A$ are direct. Hence, in this case Theorem 3.3 reads as:

*If $\sigma$ is an element of an ideal $I$, then $I$ contains every endomorphism $\alpha$ with* $\mathrm{rk}(\alpha) \leq \mathrm{rk}(\sigma)$.

It is easy to check that, for each $\kappa \geq \aleph_0$, $E_\kappa$ is an ideal of $\mathrm{End}(A)$. Write $E_0$ for $E_{\aleph_0}$, the ideal of all finite rank endomorphisms.

Now, let $I \neq 0$ be an arbitrary ideal which is properly contained in $\mathrm{End}(A)$. Since $I$ is non–trivial, there exists a non–zero endomorphism $\sigma \in I$. If $\sigma$ is of infinite rank then, obviously, $E_0 \subseteq I$. So suppose $\sigma$ is of finite rank $n \geq 1$. In this case, $I$ contains all endomorphisms of rank less than or equal to $n$. Thus if $e$ is an element of a given basis $B$, then $I$ contains the projection $\pi_e$ onto the one–dimensional subspace generated by $e$ along the subspace generated by the remaining basis elements. Therefore all finite sums $\sum_{i=1}^{k} \pi_{e_i}$ ($e_i \in B$) of such projections belong to $I$ and hence, for any $k \in \mathbf{N}$, there is an endomorphism of rank $k$ belonging to $I$. This implies that all endomorphisms of finite rank are contained in the ideal $I$ and so in either case we deduce that

$E_0 \subseteq I$. Moreover $E_{\kappa+1} \subseteq I$ whenever $\eta \in I$ for some $\eta$ of rank $\kappa$. Let $\tau+1$ be the smallest cardinal with $E_{\tau+1} \not\subseteq I$ (we may consider a successor cardinal since the ideals $E_\kappa$ form a smooth increasing chain). Then all $\eta \in I$ have rank less than $\tau$ and hence $I \subseteq E_\tau$. Also $E_\tau \subseteq I$ by the minimality of $\tau + 1$, thus $I = E_\tau$. ■

## 4. Complete modules over complete discrete valuation rings

In the last section we turn our attention to complete reduced torsion–free $R$–modules $A$ over complete discrete valuation rings $R$. Recall that the rank of a reduced torsion–free $R$–module over a complete discrete valuation ring is the rank of a basic submodule $B$ of $A$ (see Section 2). Again we define the rank of an endomorphism as the rank of its image. Moreover we call an endomorphism $\alpha$ of $A$ a *pure endomorphism* if $\mathrm{Im}(\alpha)$ is a pure submodule of $A$.

First we present a result which is similar to Theorem 3.3: for a complete reduced torsion–free $R$–module $A$ over a complete discrete valuation ring $R$ we can prove

**Theorem 4.1** *Let $I$ be an ideal of* $\mathrm{End}(A)$. *If $I$ contains a pure endomorphism $\sigma$ of rank $\kappa$ then $I$ contains all endomorphisms $\alpha$ with* $\mathrm{rk}(\alpha) \leq \kappa$.

*Proof:* Firstly we show that any endomorphism $\mu$ of $A$ is k–direct. By Lemma 2.3 it suffices to show that $\ker(\mu)$ is pure in $A$ for any $\mu \in \mathrm{End}(A)$. If $x = p^n a$ with $x \in \ker \mu$ and $a \in A$, we have $(p^n a)\mu = x\mu = 0$. Hence $p^n(a\mu) = 0$, which implies $a\mu = 0$ since $A$ is torsion–free. So,

$$p^n A \cap \ker(\mu) = p^n \ker(\mu),$$

that is, $\ker(\mu)$ is pure in $A$ and thus $\mu$ is k–direct for any $\mu \in \mathrm{End}(A)$.

Let $\sigma$ be a pure endomorphism in $I$ and assume first that $\alpha$ is a pure endomorphism of $A$. Then, by Lemma 2.3, both $\mathrm{Im}(\sigma)$ and $\mathrm{Im}(\alpha)$ are direct summands of $A$, i.e. we may write $A$ as

$$A = \mathrm{Im}(\sigma) \oplus S = \mathrm{Im}(\alpha) \oplus T.$$

If $B_\alpha$, $B_\sigma$ are basic submodules of $\mathrm{Im}(\alpha)$ and $\mathrm{Im}(\sigma)$ respectively, then there exists a direct summand $D$ of $B_\sigma$ of rank

$$\mathrm{rk}(\alpha) = \mathrm{rk}(B_\alpha) \leq \mathrm{rk}(B_\sigma) = \mathrm{rk}(\sigma),$$

which is isomorphic to $B_\alpha$. We may extend this isomorphism to an isomorphism of the completions $\mathrm{Im}(\alpha)$ and $\widehat{D}$ of $B_\alpha$ and $D$ respectively. Moreover, since $D$ is pure in $B_\sigma$, the completion $\widehat{D}$ is pure in $\mathrm{Im}(\sigma)$, hence $\widehat{D}$ is a direct summand of $\mathrm{Im}(\sigma)$ which is isomorphic to $\mathrm{Im}(\alpha)$. Thus we may apply Proposition 3.2 which implies that $\alpha \in I$. We have shown that all pure endomorphisms $\alpha$ with $\mathrm{rk}(\alpha) \leq \mathrm{rk}(\sigma)$ are contained in $I$. So, in particular, all idempotents $\pi$ with $\mathrm{rk}(\pi) \leq \mathrm{rk}(\sigma)$ belong to $I$.

Finally, let $\phi$ be any endomorphism of $A$ with $\mathrm{rk}(\phi) \leq \mathrm{rk}(\sigma)$. Then $A = \ker(\phi) \oplus C$, where $C \cong A/\ker(\phi) \cong \mathrm{Im}(\phi)$, and so $\mathrm{rk}(C) = \mathrm{rk}(\alpha)$. If $\pi$ denotes the projection onto $C$ with $\ker(\pi) = \ker(\phi)$ then $\pi \in I$ since

$$\mathrm{rk}(\pi) = \mathrm{rk}(C) = \mathrm{rk}(\phi) \leq \mathrm{rk}(\sigma).$$

Therefore $\phi = \pi\phi$ is an element of $I$. ∎

The previous theorem, however, does not characterize the ideals of $\mathrm{End}(A)$ since there are ideals which do not contain a pure endomorphism, for example, $p\,\mathrm{End}(A)$. Instead of using similar arguments as in the case of free $R$–modules we shall now use Corollary 3.4 on vector spaces to determine the ideals $I$ of $\mathrm{End}(A)$ modulo their Jacobson radicals. First we consider the Jacobson radicals $J(E_\kappa)$ of the ideals $E_\kappa$ where

$$E_\kappa = \{\eta \in \mathrm{End}(A) | \, \mathrm{rk}(\eta) < \kappa\}$$

for $\kappa \geq \aleph_0$.

**Lemma 4.2** *Let $E_\kappa$ be an ideal as defined above. Then the Jacobson radical $J(E_\kappa)$ coincides with the ideal $pE_\kappa$.*

*Proof:* First we show that $pE_\kappa = E_\kappa \cap p\mathrm{End}(A)$. Let $p\alpha \in E_\kappa$ with $\alpha \in \mathrm{End}(A)$. Then $A = A_1 \oplus \ker(p\alpha)$ and $\mathrm{rk}(A_1) < \kappa$. Since $A$ is torsion–free, $\ker(p\alpha) = \ker(\alpha)$. Hence $A\alpha = A_1\alpha$ and $\mathrm{rk}(\alpha) < \kappa$.

Thus $\alpha \in E_\kappa$. By Proposition 2.4, $p\,\mathrm{End}(A) = J(\mathrm{End}(A))$ and so it follows that

$$E_\kappa \cap p\,\mathrm{End}(A) = E_\kappa \cap J(\mathrm{End}(A)).$$

But $E_\kappa \cap J(\mathrm{End}(A)) = J(E_\kappa)$ since $E_\kappa$ is an ideal of $\mathrm{End}(A)$. Therefore

$$pE_\kappa = E_\kappa \cap p\,\mathrm{End}(A) = E_\kappa \cap J(\mathrm{End}(A)) = J(E_\kappa).$$

This completes the proof. ■

Next we will show that $E_\kappa/J(E_\kappa)$ is isomorphic to a corresponding ideal of the vector space $A/pA$ over the field $R/pR$.

**Lemma 4.3** *For any cardinal $\kappa \geq \aleph_0$, $E_\kappa/J(E_\kappa) \cong E_\kappa(A/pA)$.*

*Proof:* Every $\alpha \in E_\kappa$ induces an $R/pR$–endomorphism on $A/pA$ since $pA\alpha \subseteq pA$. So we may define a map

$$\Delta : E_\kappa \longrightarrow \mathrm{End}_{R/pR}(A/pA)$$

by

$$\alpha\Delta = \overline{\alpha} : A/pA \longrightarrow A/pA \text{ with } (a + pA)\overline{\alpha} = a\alpha + pA.$$

It is easy to check that $\Delta$ is a ring homomorphism. Moreover, the kernel of $\Delta$ is $pE_\kappa = J(E_\kappa)$. We show that $\mathrm{Im}(\Delta) = E_\kappa(A/pA)$. Certainly, $\mathrm{Im}(\Delta) \subseteq E_\kappa(A/pA)$ since the vector space rank of an endomorphism $\overline{\alpha}$ of $A/pA$ cannot be greater than $\mathrm{rk}(\alpha)$. Now let $\eta : A \longrightarrow A/pA$ and $\rho : R \longrightarrow R/pR$ be the endomorphisms defined by

$$a\eta = a + pA \text{ and } r\rho = r + pR.$$

Let us consider an endomorphism $\phi$ of $A/pA$ of rank less than $\kappa$. We can pick an $R/pR$–basis $\{x_i | \, i \in I\}$ of $A/pA$ such that $x_i\phi = 0$ for all but less than $\kappa$ of the $x_i$. Choose $y_i \in A$ such that $y_i\eta = x_i$ for all $i \in I$. Then the module generated by $\{y_i | \, i \in I\}$ is a basic submodule of $A$ by Lemma 2.1. However

$$x_i\phi = \sum_{i \in I} r_{ij}x_j,$$

where $r_{ij} \in R/pR$ with $r_{ij} = 0$ for all but finitely many $j$. Choose $s_{ij} \in R$ with $s_{ij}\phi = r_{ij}$ and $s_{ij} = 0$ whenever $r_{ij} = 0$. Finally, define $\beta : B \longrightarrow B$ by

$$y_i\beta = \sum_{i \in I} s_{ij} y_j,$$

so that $y_i\beta = 0$ for all but less than $\kappa$ of the $y_i$. Thus the unique extension of $\beta$ to an endomorphism of $A$ has rank $\mathrm{rk}(B\beta) < \kappa$ and satisfies $\beta\Delta = \phi$. Hence $\Delta : E_\kappa \longrightarrow E_\kappa(A/pA)$ is surjective, with $\ker\Delta = pE_\kappa = J(E_\kappa)$. Thus $E_\kappa/J(E_\kappa) \cong E_\kappa(A/pA)$, as required. ■

We finish the paper with the characterization of the ideals of the endomorphism ring of a complete reduced torsion–free $R$–module $A$ over a complete discrete valuation ring $R$: modulo their Jacobson radical they are characterized by a single cardinal $\kappa$.

**Theorem 4.4** *If $I$ is an arbitrary ideal of the endomorphism ring of a complete reduced torsion–free $R$–module $A$, then either $I \subseteq J(\mathrm{End}(A))$ or*

$$I/J(I) \cong E_\kappa/J(E_\kappa) \cong E_\kappa(A/pA)$$

*for some cardinal $\kappa$.*

*Proof:* Let $I$ be any ideal of $\mathrm{End}(A)$. We consider the mapping $\Delta : I \longrightarrow \mathrm{End}(A/pA)$ defined by

$$\alpha\Delta = \overline{\alpha} \text{ with } (a + pA)\overline{\alpha} = a\alpha + pA.$$

This defines a ring homomorphism with

$$\ker(\Delta) = J(I) = I \cap J(\mathrm{End}(A))$$

which is either equal to $I$ (i.e. $I \subseteq J(\mathrm{End}(A))$) or is properly contained in $I$. In the latter case $I/J(I) \cong K$ for some non–zero ideal $K$ of $\mathrm{End}(A/pA)$. Thus $K = E_\kappa(A/pA)$ for some $\kappa$ by Corollary 3.4. Therefore $I/J(I) \cong E_\kappa/J(E_\kappa)$ by Lemma 4.3. ■

## References


[1] R. Baer, Linear Algebra and Projective Geometry. Academic Press: New York, 1952.

[2] L. Fuchs, Infinite Abelian Groups, vol. I. Academic Press: New York, 1970.

[3] L. Fuchs, Infinite Abelian Groups, vol. II. Academic Press: New York, 1973.

[4] B. Goldsmith, S. Pabst and A. Scott, *Unit sum numbers of rings and modules*, to appear.

[5] I. Kaplansky, Infinite Abelian Groups. The University of Michigan, Press, 1962.

[6] H. Leptin, *Zur Theorie der überabzählbaren abelschen p–Gruppen*, Abh. Math. Sem. Univ. Hamburg **24** (1960), 79–90.

[7] W. Liebert, *Characterization of the endomorphism rings of divisible torsion modules and reduced complete torsion–free modules over complete discrete valuation rings*, Pacific J. Math. **37** (1971), 141–170.

[8] R. B. Warfield, *Homomorphisms and duality for torsion–free groups*, Math. Z. **107** (1968), 189–200.



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