Solutions to the Problems of the 36th IMO

1. First solution. Let $DN$ meet $XY$ at the point $R$. The triangles $RZD$ and $BZP$ are similar and hence $RZ/ZD = BZ/ZP$. Thus $RZ = BZ/ZD/ZP = ZX^2/ZP$. If $S$ is the point of intersection of $AM$ and $XY$, then a similar argument proves that $SZ = ZX^2/ZP$. Thus the points $R$ and $S$ coincide and the result follows.

Second solution. Choose coordinates so that the line $ABCD$ is the $x$-axis with $Z$ as origin and $XY$ is the $y$-axis. Let the coordinates of $A, B, C, D$ and $P$ be $(a, 0), (b, 0), (c, 0), (d, 0)$ and $(0, p)$, respectively. The problem can now be solved using routine calculations.

2. The expression on the left hand side of the inequality can be made a little more friendly by letting $a = 1/z, b = 1/y$ and $c = 1/z$. Then $xyz = 1$ and the inequality to be proved is:

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}$$

If $S$ denotes the left hand side, then

$$2(x + y + z)S = [(x + y) + (y + z) + (z + x)]S =$$

$$[(\sqrt{x+y})^2 + (\sqrt{y+z})^2 + (\sqrt{z+x})^2]x$$

$$[(\frac{z}{\sqrt{x+y}})^2 + (\frac{x}{\sqrt{y+z}})^2 + (\frac{y}{\sqrt{z+x}})^2] \geq (z + x + y)^2$$

by Cauchy's inequality. But the arithmetic-geometric mean inequality gives $x + y + z \geq 3$, since $xyz = 1$. Thus

$$2(x + y + z)S \geq 3(x + y + z)$$

Hence $S \geq 3/2$ and the result is proved.

3. If $A_1, A_2, A_3, A_4$ are the vertices of a square of unit area and if $r_i = 1/6$ for $i = 1, 2, 3, 4$, then the triangle $A_1 A_2 A_3$ has area $r_i + r_j + r_k$ for each triple $i, j, k (1 \leq i < j < k \leq 4)$. So the result holds for $n = 4$. 