Outline Solutions of the Problems for the 35th IMO

1. Without loss of generality \( a_1 < a_2 < \ldots < a_m \). Suppose \( a_i + a_{m+1-i} \leq n \), for some \( i \) with \( 1 \leq i \leq m \). Then \( a_i + a_{m+1-i} \leq n \), for \( j = 1, 2, \ldots \). But then the \( i \) distinct integers \( a_j + a_{m+1-i} \), \( j = 1, 2, \ldots \) must lie in the set \( \{ m, m-1, \ldots, m-i+2 \} \), which contains only \( i-1 \) elements. Thus \( a_i + a_{m+1-i} \geq n+1 \), for \( i = 1, 2, \ldots, m \). Add these inequalities to obtain the result.

2. Use coordinates. Without loss of generality, let \( M = (0, 0) \), \( B = (-1, 0) \), \( C = (1, 0) \). Let \( A = (0, a) \) and \( Q = (t, 0) \). The rest of the solution is straightforward.

3. (a) Let \( A_k \) be the set of integers in \( \{ 1, 2, \ldots, k \} \) whose base 2 representation contains exactly three 1’s and let \( g(k) \) be the number of elements in \( A_k \). Then \( f \) and \( g \) are nondecreasing functions and \( f(k) = g(2k) - g(k) \). Then

\[
f(k+1) - f(k) = g(2k+2) - g(2k) - (g(k+1) - g(k)).
\]

Now either both \( 2k+2 \in A_{2k+2} \) and \( k+1 \in A_{k+1} \) or neither is true. Thus \( f(k+1) - f(k) = 0 \) or \( 1 \), depending on whether \( 2k+1 \in A_{2k+1} \) or not. Thus \( f(k) \) does not skip any positive integer values. Since

\[
g(2^n) = \binom{n}{3} = g(2^n - 1),
\]

we get, after some calculation, \( f(2^n) = \binom{n}{2} \). Thus \( f \) is not bounded above and hence assumes every non-negative integer value.

(b) Suppose \( f(k) = m \) has a unique solution. Then

\[
f(k+1) - f(k) = 1 = f(k) - f(k-1).
\]

The former holds if and only if \( 2k+1 \in A_{2k+2} \), i.e. there are exactly two 1’s in the base 2 digits of \( k \). The same holds for \( k-1 \).

This is possible if and only if \( k-1 \) has exactly two 1’s in its base 2 representation, where the last digit is 1 and the second last digit is 0, i.e. \( k = 2^n + 2 \) for some integer \( n \geq 2 \). A calculation gives

\[
f(2^n+2) = \binom{n}{2} + 1.
\]

Thus the set of positive integers \( m \) for which \( f(k) = m \) has a unique solution is \( \{ \binom{n}{2} + 1 : n \geq 2 \} \).

4. We note that

\[
\frac{n^2 + 1}{mn - 1} + 1 = \frac{n(n^2 + m)}{mn - 1}
\]

and that

\[
\frac{m(n^2 + m)}{mn - 1} - n = \frac{m^2 + n}{mn - 1}\]

Thus \( mn - 1 \) divides \( n^2 + 1 \) if and only if it divides \( m^2 + n \) and this holds if and only if \( mn - 1 \) divides \( m^2 + 1 \).

If \( m = n \) it is easy to see that \( m = 2 \).

If \( m > n \), then \( \frac{n^2 + m}{mn - 1} = k \), an integer, implies that \( n^2 + k = \frac{m}{kn - 1} > kn^2 - n \); and thus \( (k-1)n^2 - n - k < 0 \). This implies that \( n < \frac{k}{k-1} \), if \( k > 1 \).

If \( k = 1 \), then \( n^2 + m = mn - 1 \). Thus \( m = n + 1 \). The fact that \( n - 1 \) divides 2 proves that \( n = 2 \) or 3. If \( n = 2 \), then \( m = 5 \) and if \( n = 3 \) then \( m = 5 \).

If \( k > 1 \), then \( n < \frac{k}{k-1} \) ≤ 2 implies that \( n = 1 \). Then \( m = 2 \) or 3.

Thus, if \( \frac{n^2 + 1}{mn - 1} \) is an integer, \( (m, n) \) is one of the pairs:

\[
(1, 2), (1, 3), (2, 1), (3, 1), (2, 5), (3, 5), (5, 2), (5, 3), (2, 2).
\]

It is clear that \( \frac{n^2 + 1}{mn - 1} \) is an integer if \( (m, n) \) is one of these nine pairs.

5. It is clear that \( f(x) \) can take the value 1 at most once in each of the intervals \((-1, 0)\) and \((0, \infty)\). Let \( f(a) = a \), then property (i)
implies that $f(2a + a^2) = 2a + a^2$. If $-1 < a < 0$, then $-1 < 2a + a^2 < 0$ and thus $a = 2a + a^2$. This gives the contradiction $a = 0$ or $-1$. Similarly, the assumption that $a > 0$ leads to a contradiction. Thus $f(a) = a$ implies $a = 0$. Using this fact and letting $y = x$ in (i) proves

$$x + f(x) + xf(x) = 0,$$

for all $x$ in $S$. Thus

$$f(x) = \frac{-x}{1 + x}$$

for all $x$ in $S$. It is clear that this function satisfies (i) and (ii) and is the only function with these two properties.

6. First solution. Let $A$ be the set of all positive integers of the form $q_1q_2 \ldots q_{n}$, where $q_1 < q_2 < \ldots < q_{n}$, are primes. For any infinite set $\{p_1, p_2, p_3, \ldots\}$ of primes $p_1 < p_2 < p_3 < \ldots$, we can satisfy the requirements of the problem, by taking

$$m = p_1p_2 \ldots p_{n},$$

and $n = p_2p_3 \ldots p_{n+1}$. Second solution. Let $\Pi = \{p_1, p_2, p_3, \ldots\}$ denote the set of all primes. Let

$$A_i = \{q_1q_2 \ldots q_i : q_1, q_2, \ldots, q_i \in \Pi \text{ and } p_i \not/ q_1q_2 \ldots q_i\}$$

and let $A = A_1 \cup A_2 \cup A_3 \cup \ldots$. Let $S$ be any infinite subset of $\Pi$ and let $p_k$ be in $S$. Choose distinct primes $q_1, q_2, \ldots, q_4$ in $S - \{p_k\}$. Then $m = q_1q_2 \ldots q_{k-1}q_k$ is in $A$, whereas $n = q_1q_2 \ldots q_{k-1}p_k$ is not in $A$. 

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