REMARKS ON A PROBLEM OF FINBARR HOLLAND
CONCERNING TRIGONOMETRIC POLYNOMIALS

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Let \( P_n \) denote the set of all non-negative trigonometric polynomials of degree at most \( n \), normalized to have constant term equal to 1. Thus a typical element of \( P_n \) has the form

\[
p(t) = 1 + \sum_{j=1}^{n} (a_j \cos jt + b_j \sin jt) \geq 0 \quad \text{for all real } t.
\]

A problem posed by Holland [1, Problem 4.26] essentially asks for the value of

\[
\Lambda_n = \sup_{p \in P_n} \frac{1}{2\pi} \int_0^{2\pi} (p(t))^2 \, dt.
\]

A much simpler problem is the determination of

\[
M_n = \sup_{p \in P_n} \max p(t).
\]

This was solved by Fejér [4] (or see [7; pp. 78-79]). It will be helpful to discuss this first, for it leads easily to rough bounds for \( \Lambda_n \). Fejér showed that \( M_n = n + 1 \); for a short proof see [2; §3.2]. He also showed that \( M_n \) is an attained supremum: in fact if

\[
q_n(t) = 1 + \frac{2}{n+1} (n \cos t + (n-1) \cos 2t + \cdots + \cos nt),
\]

then \( q_n \in P_n \) (for an easy calculation shows that

\[
q_n(t) = \frac{1}{n+1} \left( \sin \left( \frac{(n+1)t}{2} \right) / \sin \left( \frac{t}{2} \right) \right)^2 \geq 0 \quad (0 < t < 2\pi)
\]

and

\[
\max q_n(t) = q_n(0) = 1 + \frac{2}{n+1} (n + (n-1) + \cdots + 1) = n + 1.
\]

Goldstein and McDonald [6] observed that Fejér’s result leads to bounds on \( \Lambda_n \) as follows. If \( p \in P_n \), then

\[
\frac{1}{2\pi} \int_0^{2\pi} (p(t))^2 \, dt \leq \frac{1}{2\pi} \max p(t) \int_0^{2\pi} p(t) \, dt = \max p(t) \leq n + 1.
\]

On the other hand,

\[
\frac{1}{2\pi} \int_0^{2\pi} (q_n(t))^2 \, dt = 1 + \frac{2}{(n+1)^2} (n^2 + (n-1)^2 + \cdots + 1^2)
\]

\[
= 1 + \frac{n(2n+1)}{3(n+1)}
\]

\[
> \frac{2}{3}(n+1).
\]

Hence \( 2/3 < \Lambda_n/(n+1) < 1 \). In [6] there is further evidence favouring the conjecture that \( \Lambda_n/(n+1) \) converges to a limit \( C \in [2/3, 1] \). In fact, a proof of this conjecture, yielding the value \( C = 0.68698 \ldots \), is implicit in earlier work of Garcia, Rodemich and Rumsey [5]. In work based partly on [5], Brown, Goldstein and McDonald [2, Theorem 2] showed further that \( n(n+1)C < \Lambda_n < 1 + (n+1)C \) for all \( n \geq 1 \). Quite intricate arguments are used in both [5] and [2], and it seems worthwhile to give an elementary, self-contained, and comparatively short proof of the existence of the limit \( C \).

**Theorem.** The sequence \( \Lambda_n/(n+1) \) converges to a limit \( C \) in \([2/3, 1]\) and

\[
C = \inf_{n \geq 1} \Lambda_n/(n+1).
\]

The main step in our proof is to establish the inequality

\[
\frac{\Lambda_{nk+k-1}}{nk+k} \leq \frac{\Lambda_n}{n+1} \quad (k \geq 2, n \geq 1).
\]
Suppose for the moment that (3) is true and let $C$ be defined by (2). Fix $\epsilon > 0$ and let $N$ be such that $\Lambda_N/(N+1) < C + \epsilon$. If $n > N+1$ and $k(n)$ is the least integer such that $(N+1)k(n) > n$, then $(N+1)k(n) \leq n + N + 1$, and hence using (3) and the obvious fact that $\Lambda_n$ is non-decreasing, we obtain

$$\frac{\Lambda_n}{n+1} \leq \frac{\Lambda_{Nk(n)+k(n)-1}}{(N+1)k(n)} \frac{(N+1)k(n)}{n+1} \leq \frac{\Lambda_N}{N+1} \left(1 + \frac{N}{n+1}\right),$$

so that $\limsup \Lambda_n/(n+1) < C + \epsilon$ and hence $\Lambda_n/(n+1) \to C$.

We write

$$J(p) = \frac{1}{2\pi} \int_0^{2\pi} (p(t))^2 dt.$$

To prove (3), it suffices to show that if $p \in P_{nk+k-1}$, then

$$J(p) \leq k\Lambda_n.$$

(4)

Let such a function $p$ be given by

$$p(t) = 1 + \sum_{j=1}^{nk+k-1} (a_j \cos jt + b_j \sin jt).$$

Since $p \geq 0$,

$$0 \leq \frac{1}{2\pi} \sum_{m=1}^{k-1} \int_0^{2\pi} p(t)p(t + 2m\pi/k) dt = k - 1 + \frac{1}{2} \sum_{j=1}^{nk+k-1} \left\{ (a_j^2 + b_j^2) \sum_{m=1}^{k-1} \cos(2mj\pi/k) \right\} = k - 1 - \frac{1}{2} \sum_{j=1}^{nk+k-1} (a_j^2 + b_j^2) + \frac{1}{2} k \sum_{t=1}^{n} (a_{tk}^2 + b_{tk}^2);$$

the last-written equation follows from the fact that

$$\sum_{m=1}^{k-1} \cos(2mj\pi/k) = \begin{cases} k-1 & \text{if } k \mid j \\ -1 & \text{if } k \nmid j. \end{cases}$$

Hence

$$J(p) = 1 + \frac{1}{2} \sum_{j=1}^{nk+k-1} (a_j^2 + b_j^2) \leq k \left(1 + \frac{1}{2} \sum_{t=1}^{n} (a_{tk}^2 + b_{tk}^2)\right).$$

(5)

Note that

$$1 + \frac{1}{2} \sum_{t=1}^{n} (a_{tk}^2 + b_{tk}^2) = J(q),$$

(6)

where

$$q(t) = 1 + \sum_{t=1}^{n} (a_{tk} \cos \ell t - b_{tk} \sin \ell t).$$

(7)

If we can show that $q$ is non-negative, then we shall have $q \in P_n$, and hence $J(q) \leq \Lambda_n$. From (5) and (6) it will then follow that $J(p) \leq kJ(q) \leq k\Lambda_n$, and (4) and hence (3) will be established.

To show that $q$ is non-negative, we first associate to $p$ the harmonic polynomial $h$ defined by

$$h(re^{it}) = 1 + \sum_{j=1}^{nk+k-1} r^j (a_j \cos jt + b_j \sin jt).$$

Let $\Delta$ denote the unit disc. Since $h(e^{it}) = p(t) \geq 0$ for all $t \in [0, 2\pi]$, we have $h \geq 0$ on $\partial\Delta$ and hence, by the minimum principle, $h \geq 0$ on $\Delta$. Also define $K$ on $\Delta$ by

$$K(re^{it}) = 1 + 2 \sum_{t=1}^{\infty} r^t \cos \ell t.$$

(8)

It is easy to verify that

$$K(re^{it}) = \frac{1 - r^2}{1 - 2r \cos t + r^2} > 0 \quad (re^{it} \in \Delta).$$
(In fact, $K$ is the Poisson kernel of $\Delta$ with pole 1.) Since the series in (8) is locally uniformly convergent on $\Delta$, we have for all $r \in (0, 1)$ and all real $\theta$,
\[
0 \leq \frac{1}{2\pi} \int_0^{2\pi} h(re^{it})K(re^{i(kt+\theta)})dt \\
= 1 + \frac{1}{\pi} \sum_{l=1}^{\infty} \left( \sum_{j=1}^{\infty} r^j \int_0^{2\pi} \left( a_j \cos j\theta + b_j \sin j\theta \right) \cos (\ell kt + \ell \theta) dt \right) \\
= 1 + \sum_{l=1}^{\infty} r^{l+1} (a_l \cos \ell \theta - b_l \sin \ell \theta).
\]
Letting $r \to 1-$, we find that the function $q$ defined by (7) is indeed non-negative and, as explained earlier, (3) now follows and, therefore, $(\Lambda_n/(n+1))$ converges to the limit $C$ given by (2). The values of $\Lambda_n$, obtained from Fejér's work show that $2/3 \leq C \leq 1$.

Calculations using Mathematica and based on a representation of $\Lambda_n$ obtained by Goldstein and McDonald [6, Corollary 2] suggest the values given in the table below. I am grateful to Tony Wickstead for his help with these calculations. Our values for $\Lambda_2, \ldots, \Lambda_5$ confirm those obtained in [6, p.87], except for a small discrepancy in the value of $\Lambda_3$.

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To the best of my knowledge, the conjecture that $(\Lambda_n/(n+1))$ is decreasing remains open.

One obvious generalization of Holland's question concerns
\[
\Lambda_{n,\alpha} = \sup_{r \in \mathbb{R}} \frac{1}{2\pi} \int_0^{2\pi} (p(t))^\alpha dt \quad (\alpha > 0).
\]

References


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