Research Announcement

TAYLOR-MONOMIAL EXPANSIONS OF HOLOMORPHIC FUNCTIONS ON FRÉCHET SPACES

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Let \( \lambda := \lambda(A) \) denote a Fréchet nuclear spaces with Köthe matrix \( A \) and let \( \{E_n\}_n \) denote a sequence of Banach spaces. Let \( E := \lambda(\{E_n\}_n) := \{ (x_n)_n : x_n \in E_n \text{ and } (\|x_n\|)_n \in \lambda(A) \} \) and endow \( E \) with the topology generated by the semi-norms

\[
\|(x_n)_n\|_k := \sum_{n=1}^{\infty} a_{n,k} \|x_n\|, \quad k = 1, 2, \ldots
\]

\( E \) is a Fréchet space and \( \{E_n\}_n \) is an unconditional Schauder decomposition of \( E \). Examples of spaces which can be represented in this fashion, include all Banach spaces and all Fréchet nuclear (and some Fréchet-Schwartz) spaces with basis. Let \( H(E) \) denote the space of all \( C \)-valued holomorphic functions on \( E \) and for \( m \in N^{(N)} \), \( m = (m_1, \ldots, m_n, 0, \ldots) \) let

\[
P_m(x) = \frac{1}{(2\pi i)^n} \int_{|\lambda_i|=1} \frac{f(\sum_{i=1}^{n} \lambda_i x_i)}{\lambda_1^{m_1+1} \cdots \lambda_n^{m_n+1}} \, d\lambda_1 \cdots d\lambda_n
\]

We have

\[
f = \sum_{m \in N^{(N)}} P_m
\]

in the \( \tau_0 \), \( \tau_w \), \( \tau_d \) topologies on \( H(E) \).

The expansion (\( \ast \)) reduces to the Taylor series expansion in the case of a Banach space (i.e. if \( E_1 = E \), \( E_n = 0 \), \( n > 1 \)) and to the monomial expansion for Fréchet nuclear spaces with a basis (when \( \dim(E_n) = 1 \) for all \( n \)).

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If \((E_n)_n\) is an unconditional Schauder decomposition for the Fréchet space \(E\) then the topology of \(E\) is generated by semi-norms satisfying

\[
\left\| \sum_{n=1}^{\infty} x_n \right\| = \sup_{|\lambda| \leq 1} \left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|
\]

(\(*\))

If \((\beta_n)_{n=1}^{\infty}\) is a sequence of real numbers, \(\beta_n \geq 1\) all \(n\), let

\[
\left\| \sum_{n=1}^{j} x_n + \sum_{n=j+1}^{\infty} \beta_n x_n \right\|_{\beta,j}
\]

If \(m \in N^{(N)}\) we let \(\mathcal{P}_s(mE)\) denote the set of all \([m]\)-homogeneous polynomials on \(E\) which are homogeneous in the even variables i.e. if \(m = (m_1, m_2, \ldots, m_n, \ldots)\) then \(P \in \mathcal{P}_s(mE)\) if and only if

(i) \(P(\lambda x) = \lambda^{m_1} P(x)\) for all \(x \in E\), \(\lambda \in \mathbb{C}\).

(ii) \(P(\lambda x_2i + \sum_{n=1}^{\infty} x_n) = \lambda^{m_{2i}} P(\sum_{n=1}^{\infty} x_n)\) for all \(i\), \(x \in E\), and all \(\lambda \in \mathbb{C}\).

Our main technical tool is the following proposition.

**Proposition.** Let \((E_n)_n\) denote an unconditional Schauder decomposition for the Fréchet space \(E\), let \(F\) denote a Banach space and let \(T\) denote an \(F\)-valued linear function on \(H(E)\) which is bounded on the locally bounded subsets of \(H(E)\). Let \(\beta_n \geq 1\) all \(n\), \(\beta_{2n-1} = 2\) all \(n\) and suppose \(\sum_{n=1}^{\infty} x_n \in E\) implies \(\sum_{n=1}^{\infty} \beta_n^p x_n \in E\) for all \(p > 0\). Let \(\| \cdot \|\) denote a continuous semi-norm satisfying (\(*\)) and suppose there exists \(C > 0\) such that

\[
\|T(P)\| \leq C \|P\| \quad \text{for all } P \in \mathcal{P}_s(mE) \text{ and all } m \in N^{(N)}
\]

where \(\|P\| = \sup \{|P(x)|; \|x\| \leq 1\}\). Then, for any \(\delta > 1\), there exists \(C_1 > 0\) and a positive integer \(j\) such that

\[
\|T(P)\| \leq C_1 \delta^m \|P\|_{\beta,j}
\]