standing objection that universities (*sic*) have been over-influencing second-level mathematics syllabi. That objection is concerned with topics such as linear programming, vector analysis, calculus, even parts of trigonometry on the Lower Leaving Course; it is concerned with linear transformations, convergence of sequences and series, probability, groups, ... on the Higher Course.

The authors of the article about UCC students want the third-level sector to influence second-level syllabi. I think I know in what way, but perhaps they would spell it out? And I would like to know what recommendations they or other third-level people would make for primary level. Finally, given the composition of syllabus committees heretofore, how are all third-level interests to be represented from now on?

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SECOND INTERNATIONAL CONFERENCE

ON

HYPERBOLIC PROBLEMS

THEORY, NUMERICAL METHODS AND APPLICATIONS

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AN ESSAY ON PERFECTION

Micheál Ó Searcóid

1. ABSTRACT

A recent result of Murphy [2] concerns operators on a given infinite dimensional Hilbert space H. It states that, given a non-empty compact subset K of the complex plane, there exists a non-diagonalizable normal operator on H whose spectrum is K, if and only if K is uncountable. To effect this result, the operator theorist must use a well known result in topology, namely, that every uncountable separable metric space contains a perfect set. This excursion into topology led us to investigate how far the conditions of separability (in the metric space case this is equivalent to second countability) and metrizability can be stretched. We give below some general results about topological spaces which must contain perfect sets, and we produce a series of counterexamples to show that the conditions we have imposed on the spaces are indeed quite sharp.

2. INTRODUCTION

We recall firstly that a subset of a topological space X is said to be perfect in X if it is non-empty and is equal to the set of its limit points in X.

We now generalize the notion of compactness in the following definition: let Ω be a transfinite cardinal number and let X be a topological space. X is said to be *Ω-compact* if every open cover for X has a subcover of cardinality less than Ω. It is clear that X is compact if and only if X is $\aleph_0$-compact, and that X is Lindelof if and only if X is $\aleph_1$-compact.

In what follows, we shall be concerned also with a generalization of the concept of a G_δ. We shall consider a topological space which has the property that each singleton subset

of it can be expressed as the intersection of open sets whose number is less than Ω, where Ω is a given transfinite cardinal number. We note that any space which satisfies such a property is necessarily a T_1 space. Also, when $\Omega = \aleph_1$ the property is equivalent to that of each singleton set being a G_δ. We note also that every singleton set in a first countable T_1 space is necessarily a G_δ; and further that a compact Hausdorff space has the property that each of its singleton subsets is a G_δ if and only if it is first countable.

For all other topological definitions, we follow Kelley [1].

3. TOPOLOGICAL RESULTS

We begin with an elementary lemma, which mimics the compact case:

3.1 LEMMA. *Let Ω be a transfinite cardinal number and let X be an Ω-compact topological space. Let Y be a closed subspace of X. Then Y is also Ω-compact.*

PROOF. Suppose $(U_\alpha)_{\alpha \in A}$ is an open cover for Y, where A is some indexing set. Then, for each $\alpha \in A$, $U_\alpha = V_\alpha \cap Y$ for some set V_α open in X. Then $X \setminus Y \cup (V_\alpha)_{\alpha \in A}$ is an open cover for X, so there exists a subset B of A with $\text{card}(B) < \Omega$ such that $X \setminus Y \cup (V_\alpha)_{\alpha \in B}$ covers X. Then $(U_\alpha)_{\alpha \in B}$ covers Y.

We are now ready to prove our main theorem:

3.2 THEOREM. *Let T and Ω be transfinite cardinal numbers with $T \leq \Omega$. Let X be a regular topological space which is T-compact and in which every singleton subset can be expressed as the intersection of $\leq \Omega$ open sets. Suppose $\text{card}(X) \geq \Omega$; then there exists an T-compact subset of X which is perfect in X.*

PROOF. Let $T = \{x \in X: \forall N \in \text{Nbd}(x), \text{card}(N) \geq \Omega\}$. We shall show that T is perfect in three stages.

(a) Firstly we show that if Y is any closed subspace of X with $\text{card}(Y) \geq \Omega$, then $Y \cap T \neq \emptyset$. Indeed, suppose, on the contrary, that $Y \cap T = \emptyset$; then, for each $y \in Y$, there exists an open neighbourhood N_y of y with cardinality less than Ω. Now Y is T-compact by 3.1, so there is a subset V of Y of cardinality less than T such that

$$Y \subseteq \bigcup_{y \in V} N_y .$$

This leads to the contradictory conclusion that $\text{card}(Y) < \Omega$. Hence we have $T \cap Y \neq \emptyset$; in particular, T is not empty.

(b) Secondly, T is closed in X, for, if x is any element of X and if N is a neighbourhood of x which has non-empty intersection with T, then N is a neighbourhood of some point of T and hence has cardinality not less than Ω. So T is closed in X, and T is T-compact by 3.1.

(c) Thirdly we show that every point of T is a limit point of T. Let $t \in T$, and suppose that K is a closed neighbourhood of t; then

$$\text{card}(K \setminus \{t\}) \geq \Omega .$$

Let $(N_i)_{i \in I}$ be a family of open sets of X with $\text{card}(I) < \Omega$ which satisfies

$$\{t\} = \bigcap_{i \in I} N_i .$$

Then

$$K \setminus \{t\} = \bigcup_{i \in I} (K \setminus N_i)$$

Hence we have $\text{card}(K \setminus N_i) \geq \Omega$ for some $i \in I$.

Part (a) of the proof now allows us to deduce that

$$T \cap (K \setminus \{t\}) \neq \emptyset .$$

Since X is a regular space, this is sufficient to conclude that t is a limit point of T.

Hence T is perfect in X.

Some notes regarding the separation property of our topological space are in order. Firstly, a T_1 space is regular and compact if and only if it is compact Hausdorff. Secondly, a regular Lindelof space is necessarily normal. These considerations account for the formulation of the following two special cases of our theorem:

3.3 COROLLARY (i). *Let X be a first countable compact Hausdorff space. If X is uncountable, then X contains a perfect set.*

(ii) *Let X be a normal Lindelof space in which every singleton set is a G_δ. If X is uncountable, then X contains a perfect set.*

PROOF. This is precisely what theorem 3.2 says when

(i) $T = \aleph_0$ and $\Omega = \aleph_1$.

(ii) $T = \Omega = \aleph_1$.

It is well known that a compact Hausdorff space is metrizable if and only if it is second countable. It is well to remind the reader at this point that there do exist first countable compact Hausdorff spaces which are not second countable. For an example, consider $[0,1] \times [0,1]$ in the order topology induced by dictionary order: $(a,b) < (c,d)$ means that either $a < c$ or both $a = c$ and $b < d$.

Actually, since every second countable space is hereditarily Lindelof, the general theorem for these spaces is much more easily stated:

3.4 THEOREM. *Let X be an uncountable second countable topological space. Then X contains a perfect set.*

PROOF. Let $T = (x \in X: \forall N \in \mathrm{Nbd}(x), N \text{ is uncountable})$. Since every subspace of X is Lindelof, an argument similar to that of 3.2(a) shows that T has non-empty intersection with every uncountable subset of X. So $X \setminus T$ is countable, and it follows immediately that every point of T is a limit point of T. That T is closed is proved as in 3.2(b). Hence T is perfect and the result is proven.

We should like to investigate now the necessity or otherwise of the toplogical conditions which we placed on the space X in theorem 3.2 in order to ensure a successful outcome. The following three examples are instructive:

3.5 EXAMPLE. Let X be any set with the discrete topology. Then X is Hausdorff since all subsets are clopen; X is first countable since each singleton subset is open; and X is locally compact since each singleton set is clopen and compact. Yet, whatever its cardinality, X contains no perfect set because all its points are isolated.

3.6 EXAMPLE. Let Y be an infinite set and let $y \in Y$. We define a toplogy on Y by declaring as open each set whose complement is finite or whose complement contains y. Then Y is Hausdorff since there is only one singleton set in Y which is not clopen; Y is compact since each open cover for Y contains a neighbourhood of y, which necessarily has finite complement; yet, however large the set Y is, Y contains no perfect set, since only one of its points is not isolated.

3.7 EXAMPLE. Let Z be any non-empty set and let $z \in Z$. We define a topology on Z by declaring as open each subset of Z which does not contain z, and Z itself. Then Z is compact since every open cover for Z contains the only neighbourhood of z, namely Z; Z is first countable since z has exactly one neighbourhood and every singleton set other than (z) is open.

Z is a T_0 space for the same reason. Yet Z, regardless of its cardinality, contains no perfect set since only one of its points is not isolated.

This last example is not as satisfying as the other two. Although Z is T_0, it is not T_1, and although it is first countable, not every singleton set can be expressed as the intersection of open sets. Ideally we should have liked to find an uncountable compact first countable T_1 space which contains no perfect set.

Example 3.5 shows us that the essential role played by T-compactness in the proof of 3.2 cannot be assumed by any local property. It is true, however, that a local property is sufficient to provide us with a converse of the most special case of our theorem:

3.8 THEOREM. *Let X be a locally compact Hausdorff space. If X contains a perfect set P, then every set in the relative topology of P, other than $\emptyset$, is uncountable.*

PROOF. Suppose P is a perfect set in X, and let U be an open set in X which has non-empty intersection with P. Then $\overline{U \cap P}$ is closed in X so is locally compact. Now, $\overline{U \cap P}$ is a T_1 space so each of the sets $(U \cap P) \setminus \{t\}$ $(t \in U \cap P)$ is open in $\overline{U \cap P}$. Furthermore, since P is perfect, each of these sets is also dense in $\overline{U \cap P}$. Now, $\overline{U \cap P}$ is a locally compact regular space so that Baire's theorem holds; hence $\bigcap_{t \in I} (U \cap P) \setminus \{t\}$ is dense in $\overline{U \cap P}$ for any countable subset I of $U \cap P$. It follows that $U \cap P$ is uncountable.

We are now in a position to state a necessary and sufficient condition for a certain type of topological space to contain a perfect set. Moreover, we can identify that part of the space in which perfect sets must lie:

3.9 THEOREM. *Let X be a first countable compact Hausdorff space. Then X contains a perfect set if and only if X is uncountable. In that case, every perfect set in X is contained in T, where T = {x ∈ X: every neighbourhood of x is uncountable}.*

PROOF. By 3.2 and 3.8.

Of course it is not true that all perfect sets are uncountable. The most primitive counterexample to that conjecture is an indiscrete space of two elements. This is compact, first countable and perfect in itself. A more formidable counterexample would be any countably infinite set with the finite complement topology. This has all the above properties and is a T_1 space besides.

We have proved a converse to 3.3(i). Any attempt to produce a converse in general to the main theorem is, however, doomed to failure. In fact, the converse to our second special case is easily seen to be false. The set of rational numbers with the usual metric gives us a normal first countable T_1 space, as all metric spaces do, which is Lindelof since it is countable. This set is clearly perfect in itself, yet is not uncountable.

This last counterexample brings to prominence that perpetual defect of the rational numbers - that they are incomplete. Complete metric spaces behave well in the present context. Indeed, theorem 3.8 has a companion theorem, proved in exactly the same way:

3.10 THEOREM. *Let X be a complete metric space. If P is a perfect set in X, then every set in the relative topology of P, except for $\emptyset$, is uncountable.*

It should be added that complete metric spaces do not necessarily contain perfect sets even when they have large enough cardinality. Indeed, any set can be endowed with the discrete metric to produce a complete metric space; as we have already

noted, no discrete space contains a perfect set.

In metric spaces, we can look for perfect sets which are small in the sense of the metric. Our main theorem yields us a result:

3.11 THEOREM. *Let Ω be a transfinite cardinal number. Let X be an Ω-compact metric space and suppose* card(X) $\geq \Omega$. *Let ε be a positive real number; then there exists an Ω-compact subset of X which is perfect in X and whose diameter is not greater than ε.*

PROOF. The open balls $\{x \in X: \text{dist}(x,a) < \varepsilon\}$ $(a \in X)$ cover X; therefore some subset of them of cardinality less than Ω also covers X. Since card(X) $\geq \Omega$, it follows that at least one of the balls, say B, has cardinality not less than Ω. Now $\overline{B}$ is Ω-compact by 3.1; being a metric space, $\overline{B}$ is also T_3 and first countable. Theorem 3.2 now furnishes us with an Ω-compact perfect set in $\overline{B}$, which is also of course Ω-compact and perfect in X. Its diameter does not exceed ε.

This leads us to a very special case indeed, where we can say a little more:

3.12 THEOREM. *Let X be a subspace of $\mathbb{R}^n$, where n is a natural number. Let ε be a positive real number. We have:*

(a) If X is uncountable, then X contains a perfect set of diameter not more than ε; further, if X is closed, then this perfect set is both perfect in $\mathbb{R}^n$ and compact in $\mathbb{R}^n$.

(b) If X contains a perfect set, then the closure $\overline{X}$ of X in $\mathbb{R}^n$ is uncountable.

PROOF. (a) Since every subspace of $\mathbb{R}^n$ is Lindelof, the first part is given by 3.11. The resulting perfect set P in X certainly has no isolated point in $\mathbb{R}^n$ and, if X is closed, it is closed in $\mathbb{R}^n$, hence perfect in $\mathbb{R}^n$. Since P is bounded it is also compact by the Heine-Borel theorem.

(b) If P is perfect in X, then it is clear that $\overline{P}$ is perfect in $\overline{X}$. Since $\overline{X}$ is locally compact, the result follows from 3.8.

REFERENCES

1. KELLEY, J.L.
'General Topology', Van Nostrand, 1955.

2. MURPHY, G.J.
"Diagonalizing Operators", (to appear) 1986.

Roinn na Matamaitice,
Colaiste na hOllscoile,
Corcaigh,
Eire.