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THE USE OF BERNSTEIN POLYNOMIALS IN CAD/CAM .. BEZIER CURVES

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INTRODUCTION

In this article we present an application of the Bernstein approximation theorem to CAD/CAM (Computer Aided Design/Computer Aided Manufacture) and to Computer Graphics. In these fields we need to be able to model complex shapes. Indeed, the method was originally proposed by P. Bezier and was used to model surfaces in automobile design for the French firm Regie Renault (cf. [1]). Many other techniques are available for modelling curves and surfaces such as B-splines, cubic splines, standard polynomial interpolation, parabolic blending etc. A useful book on these topics is [2] (with code in BASIC). We include Bezier curves here for two main reasons: first, their practical value, and, second, their roots in classical approximation theory.

MATHEMATICAL BACKGROUND

We start with the Weierstrass approximation theorem, which states roughly that one can approximate a function by a polynomial. Note that the theorem is non-constructive in the sense that neither the statement of the theorem nor its proof allows us to construct the polynomial. The result goes as follows.

Assume that $f(t)$ *is a continuous function on the interval* $[0,1]$. *Given* $\varepsilon > 0$, *there exists an integer* $N > 0$ *and a polynomial* $P(t)$ *of the same degree such that*

$$|f(t) - P(t)| < \varepsilon \quad \text{for all} \quad t \in [0,1]. \tag{1}$$

This result forms the basis of much numerical analysis, e.g. numerical integration and interpolation, finite element analysis etc. We note that the polynomial P which satisfies (1) is not necessarily of interpolating type.

Bernstein (1912) actually constructed a polynomial which satisfied the conditions of the theorem. The result is:

Let $f(t)$ *be a continuous function on the interval* $[0,1]$. *Define the* nth *degree polynomial* $P(f;t)$ *by*

$$P(f;t) = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} t^j (1-t)^{n-j} f(j/n). \qquad (2)$$

Then the polynomials $P(f;t)$ *converge uniformly on* $[0,1]$; *this means that given* $\varepsilon > 0$, *there exists an integer* N *such that for all* $n \geq N$ *we have*

$$|f(t) - P(f;t)| < \varepsilon \quad \textit{for all} \quad t \in [0,1]. \qquad (3)$$

For a proof of this result, see [3].

BEZIER CURVES

In representation (2) we define the so-called *control points*

$$p_j = f(j/n) \quad \text{for} \quad j = 0, \ldots, n, \qquad (4)$$

and the so-called *blending functions*

$$B_{j,n}(t) = \frac{n!}{j!(n-j)!} t^j (1-t)^{n-j}. \qquad (5)$$

In this case we can write the resulting curve as a polynomial of degree n as follows:

$$P(t) = \sum_{j=0}^{n} p_j B_{j,n}(t). \qquad (6)$$

Notice that (6) is a vector equation: let $p_j = (x_j, y_j, z_j)$, $j = 0, \ldots, n$, be the coordinates of the control vertices and suppose that $P(t) = (x(t), y(t), z(t))$. Then from (6) we have

the following:

$$x(t) = \sum_{j=0}^{n} x_j B_{j,n}(t)$$

$$y(t) = \sum_{j=0}^{n} y_j B_{j,n}(t)$$

$$z(t) = \sum_{j=0}^{n} z_j B_{j,n}(t)$$

These last three equations form the basis for any computer implementation of Bezier curves. Input for such a program would be the control points and the number of subdivisions of the interval $0 \leq t \leq 1$. This last parameter will basically determine the number of points on the newly generated Bezier curve. In most cases the code would be written in FORTRAN.

REMARKS

1. We can think of a Bezier curve as being associated with the vertices of a polygon which uniquely define the curve slope. Only the first and last vertices of the polygon actually lie on the curve; however, the other vertices define the derivatives, order and shape of the curve (see Fig. 1).

2. By changing the control vertices we can change the resulting Bezier curve and this property gives a good intuitive feeling for the CAD/CAM designer.

3. The number of polygon vertices fixes the order of the resulting polynomial which defines the curve and furthermore, the Bernstein basis has a global span, i.e. the values of the blending functions given by (5) are nonzero for all parameter values over the entire span of the curve. Thus, changing a control vertex changes the entire curve. This eliminates the possibility of producing local change. These problems can be overcome but one must resort to the so-called B-splines (cf. [4].

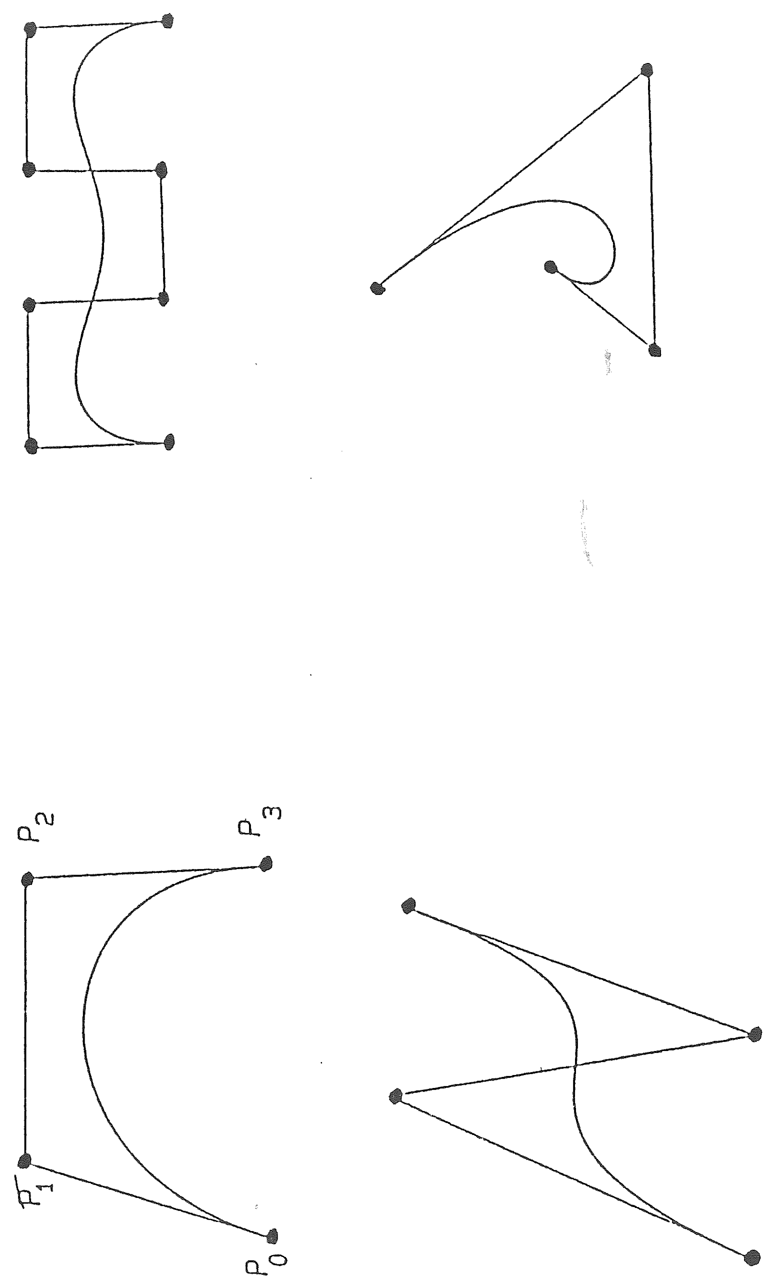

FIGURE 1: Some Bezier Curves and Associated Control Vertex Polygons

BEZIER SURFACES

Equation (6) can be generalized to three-dimensional surfaces by generating the Cartesian product of two Bezier curves. The resulting Bezier surface can be written as (again, a vector equation):

$$P(t,s) = \sum_{i=0}^{n} \sum_{j=0}^{m} p_{ij} B_{i,n}(t) B_{j,m}(s) \qquad (8)$$

In this case we have to input (n+1) x (m+1) control points (p_j). For an implementation of these surfaces, see [2], pp. 230-231.

CONCLUSION

We are only able to give a short review of one topic in a fast growing area. Many other techniques exist for approximating curves and surfaces and new methods are being constantly developed. For a good introduction to Computer Graphics, see [4], in particular pp. 309-331.

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A CONNECTED TOPOLOGY WHICH IS NOT LOCALLY CONNECTED

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To demonstrate that a connected topological space may not be locally connected, authors of modern text-books on point-set topology usually employ an example, either of a geometrical nature in the real plane (such as the so-called "topologist's sine curve". and "infinite broom"), or of a number theoretical nature in the integers (such as the "relatively prime integer topology"), or of an analytical nature in the real line (such as the "indiscrete or pointed extensions of the reals" and the "one-point compactification of the rationals"; see [1]). The complete exposition of such an example tends to rely heavily on a knowledge of the various intrinsic properties of the supporting set. For the instructor who may, perhaps, prefer a more abstract and topologically succinct example, an alternative is readily available.

Let X be an infinite set containing distinct points x,y. A topology τ for X may be defined by declaring open, apart from $\emptyset$ and X itself, those subsets G of X for which $y \notin G$ *and* either $x \notin G$ or X-G contains (at most) a finite number of points. Observe that $\tau = (\gamma \cup \varepsilon(x)) \cap \varepsilon(y)$, where γ denotes the well known cofinite topology for X and $\varepsilon(x)$, $\varepsilon(y)$ denote, respectively, the excluded point topologies $\{G \subseteq X : x \notin G\} \dot{\cup} \{X\}$ and $\{G \subseteq X : y \notin G\} \cup \{X\}$ (see [1]). That is, τ is the intersection of a Fort topology $\gamma \cup \varepsilon(x)$ and an excluded point topology $\varepsilon(y)$.

It is immediate that (X,τ) is a connected space (since $\tau \subseteq \varepsilon(y)$ and $(X,\varepsilon(y))$ is obviously a connected space). Let U be any proper τ-open neighbourhood of x. Thus $y \notin U$ and X-U is finite. If $z \in U$, $z \neq x$, then $\{z\}$ and $U-\{z\}$ are each τ-open (since y belongs to neither, $x \notin \{z\}$ and $X-(U-\{z\}) = (X-U) \cup \{z\}$ is finite) so that U is not τ-connected. It