ON THE EXISTENCE OF MAXIMAL LOWER BOUNDS

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A *lower semilattice* is a partially-ordered set in which each two elements possess a maximum lower bound or infimum, and a routine induction argument shows that in such a system every finite set also possesses an infimum. Many partially-ordered sets, of course, fail to behave so nicely. To mention one classic example, we can impose a natural partial order on the four dimensional space-time continuum of special relativity by saying that one 'event' (x,y,z,t) precedes another (x',y',z',t') whenever light from the first could reach the 'place' of the second at or before the 'time' of the second, thus:

$$(x,y,z,t) \leq (x',y',z',t') \iff$$

$$\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \leq c(t'-t)$$

where the positive constant c represents the speed of light. It can be shown that in this structure the set of lower bounds of two events (their "common history", so to speak) never possesses a maximum element, except in the trivial case where one of the events precedes the other. There is, however, in this example and in many others, an abundance of *maximal* lower bounds: the common history of two events is 'inductive' in the sense described below. This note arises from an investigation of maximal lower bounds for two or more elements; in particular it concerns the failure of the analogue of the result referred to in the first sentence above: inductiveness of the set of lower bounds for each *two* elements does not imply the same property for *three*.

Following Birkhoff [2] let us call a non-null subset A of a partially-ordered set $(E,\leq)$ *inductive* when to each point x of A there corresponds a maximal point m of A satisfying

$x \leq m$. To avoid possible confusion we should point out immediately that this meaning of the term differs from that of Bourbaki ([3], page 154), who applies it to subsets A of a partially-ordered set such that every chain in A has an upper bound in A. Of course, Zorn's lemma readily shows such a set to be inductive in the Birkhoff sense. To disprove the converse, take E as the set of points (x,y) in the coordinate plane satisfying both $-1 < x-y < 1$ and $x+y \leq 1$, with the coordinatewise partial order described by

$$(x,y) \leq (x',y') \text{ if and only if } x \leq x' \text{ and } y \leq y'.$$

Then E (as a subset of itself) is "Birkhoff-inductive" since each of its members (x,y) lies under the maximal member $((1+x-y)/2,\ (1-x+y)/2)$, but is not "Bourbaki-inductive" since the chain $\{(x,y) \in E : x = 0\}$ has no upper bound in E.

Now consider the following condition $\mu_L(\alpha)$, applicable to $(E,\leq)$, α denoting a cardinal number:

> for each non-null subset B of E having cardinality at most α, the set $L(B)$ of all its common lower bounds is non-null and inductive. .. $\mu_L(\alpha)$

We term $(E,\leq)$ a $\mu_L(\alpha)$*-system* if it satisfies this condition. If α and β are two cardinal numbers satisfying $\alpha < \beta$ then it is immediate that $\mu_L(\beta)$ implies $\mu_L(\alpha)$; we here exhibit an example to show that in general the converse implication is never valid: that is, that the conditions $\mu_L(\alpha)$ are all logically distinct.

<u>PROPOSITION 1</u>. *Let $\alpha \geq 2$ be a given cardinal number. There is a partially-ordered set satisfying condition $\mu_L(\alpha')$ for every cardinal number α' less than α, but not satisfying condition $\mu_L(\alpha)$.*

PROOF. Denote the set of positive integers by N. Take an index set A of cardinality α, a set D of cardinality $\alpha . \aleph_0$ comprising the distinct elements d_n^a ($a \in A$, $n \in N$), a set X of cardinality α comprising the distinct elements x^a ($a \in A$), and a set C of cardinality $\aleph_0$ comprising the distinct elements c_n ($n \in N$), where D, X and C are pairwise-disjoint. On the set $E = C \cup D \cup X$ we define a partial order by specifying, as follows, the strict lower bounds of each of three typical elements c_n, d_n^a and x^a:

(i) $c_n >$ each of $c_1, c_2, \ldots, c_{n-1}$, while c_1 is minimum in E;

(ii) $d_n^a >$ each of $c_1, c_2, \ldots, c_n$;

(iii) $x^a > d_n^e$ for all n and for all $e \in A \setminus \{a\}$ and also c_n for all n.

It may be helpful to refer to Fig. 1, which is a diagrammatic representation of this construct in the case where $\alpha = 3$. Note in particular that

$$\text{if } z \in C \cup D \text{ then } L(z) \text{ is finite.} \qquad (*)$$

Now Let α' be a cardinal less than α, and B a non-null subset of E whose cardinality β is at most α'; three cases may arise:

(I) $B \subseteq X$: then by (*), L(B) is finite, and therefore inductive.

(II) $B \subseteq X$ and $\beta = 1$, that is, $B = \{x^a\}$ for some a: then $L(B) = L(x^a)$ is trivially inductive.

(III) $B \subseteq X$ and $2 \leq \beta \leq \alpha' < \alpha$: then $L(B) \subseteq C \cup D$ and there is $e \in A$ such that $x^e \in X \setminus B$. Hence $d_n^e \in L(B)$ for all $n \in N$; so while each member of $D \cap L(B)$ is maximal in L(B), each c_n in L(B) lies under the maximal member d_n^e of L(B), and L(B) is again inductive.

The condition $\mu_L(\alpha')$ is therefore satisfied. On the other hand, $\mu_L(\alpha)$ is not: for X is a subset of E having cardinality

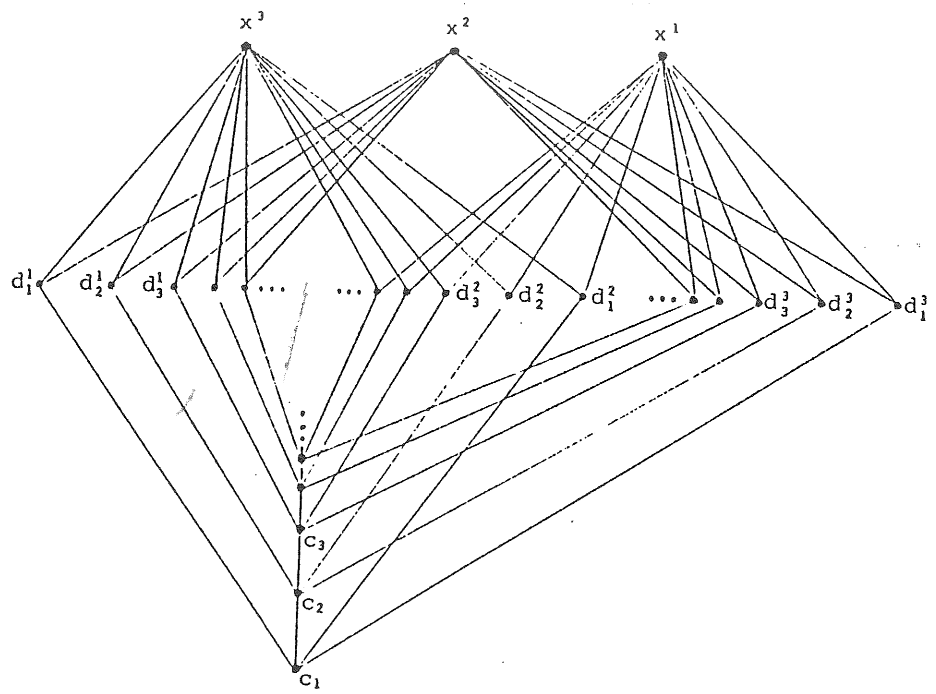

FIGURE 1

α, and $L(X) = C$ has no maximal element.

We can, however, obtain further positive connections between the conditions $\mu_L(\alpha)$ either by imposing some additional condition on the partial ordering, or by insisting that it be compatible with a suitable topology. Note the following definitions: $(E, \leq)$ is *down-directed* if each two of its elements have at least one common lower bound, a subset of $(E, \leq)$ is called *diverse* if no two elements of it are commensurable, a *decreasing* subset D of E is one for which $x \leq y$ and $y \in D$ together imply $x \in D$, and a partially-ordered topological space $(E, \leq, \tau)$ is termed T_1-*ordered* [4] if, for each of its elements x, both L(x) and the set M(x) of all the upper bounds of x are closed sets.

LEMMA. *If $(E,\leq)$ has no infinite diverse subsets, then its inductive decreasing subsets are precisely the finite unions of sets of the form $L(x)$.*

PROOF. Clearly such a set is inductive decreasing. For the rest, it suffices to note that if D is an inductive decreasing subset, then $D = \bigcup\{L(m) : m \in M\}$ where M is the set of maximal points of D; and that M, being diverse, is here finite.

PROPOSITION 2. *A $\mu_L(2)$-system without infinite diverse subsets is a $\mu_L(n)$-system for every positive integer n.*

PROOF. For each pair of points x,y in the $\mu_L(2)$-system $(E,\leq)$ denote by $\eta(x,y)$ the set of maximal lower bounds of x and y, so that

$$L(x,y) = \bigcup\{L(m) : m \in \eta(x,y)\}.$$

Observe that for any z in E we have

$$L(x,y,z) = \bigcup\{L(n) : n \in \eta(m,z),\ m \in \eta(x,y)\};$$

and if each of the (diverse) sets $\eta(m,z)$, $\eta(x,y)$ is finite, this (by the lemma) is inductive and $(E,\leq)$ is a $\mu_L(3)$-system. The obvious induction extends this argument to establish the proposition.

PROPOSITION 3. *Let $(E,\leq,\tau)$ be down-directed, compact and T_1-ordered; then it is a $\mu_L(\alpha)$-system for every cardinal $\alpha \geq 1$.*

PROOF. Let B be a non-null subset of E. The family $\{L(b) : b \in B\}$ of closed subsets of compact E has the finite intersection property, and therefore

$$\emptyset = \bigcap\{L(b) : b \in B\} = L(B)$$

(compare [6], Theorem 1). Let z be any point of $L(B)$, and C a chain in $M(z) \cap L(B)$; again, the family $\{M(c) \cap L(B) : c \in C\}$ of closed subsets of (closed, and therefore compact) $M(z) \cap L(B)$

has the finite intersection property, whence C has an upper bound in $M(z) \cap L(B)$; an application of Zorn's lemma now shows that $M(z) \cap L(B)$ has a maximal point, which is then maximal in $L(B)$ and lies over z: thus $L(B)$ is inductive.

REMARKS

Bearing in mind the power and the widespread use of maximality arguments in many areas of mathematics, there are surprisingly few references in the literature to the ideas here presented. The only major investigation seems to be that of Benado (see, e.g. [1]) who explored in detail what we have here termed $\mu_L(2)$-systems (without the assumption of down-directedness) but not $\mu_L(3)$ or beyond. The present writer's involvement is due to an attempt to generalize the idea of a *topological semilattice* - by which is meant a semilattice equipped with a topology such that the map taking each pair of elements to their infimum is continuous. If in an arbitrary down-directed partially-ordered topological space one considers continuity of the map taking each n-tuple of points to the set of *all* their common lower bounds, having first made a sensible choice of topology for the ranges of these maps, one gets a hierarchy of conditions (for varying n) each of which specializes to "continuity of the infimum" in the semilattice case. It transpires (see [5]) that the conditions $\mu_L(n)$ are convenient for obtaining satisfactory product theorems concerning such bound-continuity conditions. The exploration of these conditions is still incomplete: for example, no full understanding of when a sub-(order/topological)-system inherits them has been obtained, and $\mu_L(n)$-systems may well have a role to play in this matter also.

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THE USE OF BERNSTEIN POLYNOMIALS IN CAD/CAM .. BEZIER CURVES

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INTRODUCTION

In this article we present an application of the Bernstein approximation theorem to CAD/CAM (Computer Aided Design/Computer Aided Manufacture) and to Computer Graphics. In these fields we need to be able to model complex shapes. Indeed, the method was originally proposed by P. Bezier and was used to model surfaces in automobile design for the French firm Regie Renault (cf. [1]). Many other techniques are available for modelling curves and surfaces such as B-splines, cubic splines, standard polynomial interpolation, parabolic blending etc. A useful book on these topics is [2] (with code in BASIC). We include Bezier curves here for two main reasons: first, their practical value, and, second, their roots in classical approximation theory.

MATHEMATICAL BACKGROUND

We start with the Weierstrass approximation theorem, which states roughly that one can approximate a function by a polynomial. Note that the theorem is non-constructive in the sense that neither the statement of the theorem nor its proof allows us to construct the polynomial. The result goes as follows.

Assume that f(t) *is a continuous function on the interval* [0,1]. *Given* $\varepsilon > 0$, *there exists an integer* $N > 0$ *and a polynomial* P(t) *of the same degree such that*

$$|f(t) - P(t)| < \varepsilon \quad \text{for all} \quad t \in [0,1]. \qquad (1)$$

This result forms the basis of much numerical analysis, e.g. numerical integration and interpolation, finite element anal-