BANACH SPACE ULTRAPRODUCTS

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INTRODUCTION

This note presents a useful tool in Banach space theory: *ultraproducts of Banach spaces*. These provide a uniform method for manufacturing locally similar Banach spaces. In this way they relate local (finite dimensional) and global (infinite dimensional) structure.

Prerequisites are in Section 1; Section 2 contains definitions. Section 3 sketches some typical applications in the local theory of Banach spaces. The conclusion mentions other areas in which ultraproducts are profitably employed. Results for which no reference is given can be found in [4] and [1] which include bibliographies.

1. FILTERS AND ULTRAFILTERS

To set up and handle ultraproducts of Banach spaces effectively, one requires some basic facts about filters and ultrafilters on sets.

Let I be a non-empty set and $\mathcal{P}(I)$ be the power set of I. A *filter* on I is a subset F of $\mathcal{P}(I)$ such that:

F1 $\emptyset \notin F$.

F2 $A \in F$, $B \in F$ imply $A \cap B \in F$.

F3 $A \in F$, $A \subset B \subset I$ imply $B \in F$.

An *ultrafilter* on I is a maximal (proper) filter on I. Equivalently, U is an ultrafilter on I iff (1) U is a filter on I and (2) for all $x \in \mathcal{P}(I)$, $X \in F$ iff $I - X \notin F$.

A trivial application of Zorn's lemma shows that every filter on I can be extended to an ultrafilter on I.

The following topological property of ultrafilters forms the basis of the definition of the Banach space ultraproduct.

THEOREM 1.1. *Let K be a compact Hausdorff topological space; let I be a non-empty set and U be an ultrafilter on I. Then, for each family $(x_i)_{i \in I}$ in K, there exists a unique point $x \in K$ such that, for every neighbourhood V of x,*

$$\{i \in I : x_i \in V\} \in U.$$

The point x is called the *limit of* $(x_i)_{i \in I}$ *with respect to* U, and is denoted by $\lim_U x_i$.

2. ULTRAPRODUCTS OF BANACH SPACES

Let $\{(E_i, \|\ \|) : i \in I\}$ be a family of Banach spaces over $\mathbb{C}$ (or $\mathbb{R}$) indexed by the set I. U is an ultrafilter on I.

Define Π_0 and N_U as follows:

$$\Pi_0 := \{(x_i)_{i \in I} : x_i \in E_i, \sup_{i \in I} \|x_i\| < \infty\}$$

$$N_U := \{(x_i)_{i \in I} : (x_i)_{i \in I} \in \Pi_0, \lim_U \|x_i\| = 0\}.$$

Note that, for $(x_i)_{i \in I} \in \Pi_0$, $\lim_U \|x_i\|$ exists and is unique by theorem 1.1.

Let $\|\ \|$ be the supremum norm on Π_0:

$$\|(x_i)_{i \in I}\| := \sup_{i \in I} \|x_i\|.$$

Then $\ell_\infty((E_i)_{i \in I})$ is the Banach space $(\Pi_0, \|\ \|)$ over $\mathbb{C}$. It is easy to check that N_U is a closed subspace of $\ell_\infty((E_i)_{i \in I})$.

The *ultraproduct* of the family $((E_i, \|\ \|) : i \in I)$ modulo U is the quotient space $\ell_\infty((E_i)_{i\in I})/N_U$ with the canonical quotient norm, and is denoted $(E_i)_U$ or $\prod_{i\in I}{}_0 E_i/U$. $(E_i)_U$ is called a *Banach space ultraproduct*; in the case where $E_i = E$ for all $i \in I$ $(E)_U$ is also written E^I/U and is termed the *Banach space ultrapower* of E modulo U.

It is convenient and customary to denote elements of $(E_i)_U$ by $(x_i)_U$ so that

$$(x_i)_U := (x_i)_{i\in I} + N_U.$$

Notice that the quotient norm on $(E_i)_U$ is given by the equation:

$$\|(x_i)_U\| := \inf_{n\in N_U} \|(x_i)_{i\in I} + n\| = \lim_U \|x_i\|.$$

For each ultrapower E^I/U of E there is a canonical isometric embedding $\bar{i}$ of E into E^I/U:

$$\bar{i}(x) := (x_i)_U \quad \text{where } x_i = x \text{ for all } i \in I$$

$$\|\bar{i}(x)\| = \lim_U \|x_i\| = \|x\|.$$

If E is finite dimensional, then E and E^I/U are isometrically isomorphic. The closed balls of E are compact so that for every bounded family $(x_i)_{i\in I}$ in E the limit $\lim_U x_i$ exists in E (by theorem 1.1) and $\|\lim_U x_i\| = \lim_U \|x_i\|$ so that the map $(x_i)_{i\in I} \to \lim_U x_i$ is a linear surjection with kernel N_U, hence induces an isometric isomorphism of E^I/U and E.

The following proposition introduces the theme of the structure-preserving properties of ultraproducts.

PROPOSITION 2.1. *The following classes of Banach spaces are closed under ultraproducts:*

(i) Banach algebras;

(ii) C algebras;*

(iii) C(K)-spaces;

(iv) L^p spaces.

The class of JB triple systems is closed under ultrapowers.*

PROOF. To prove (i) and (ii) define the natural multiplication and involution on $(E_i)_U$:

$$(x_i)_U \cdot (y_i)_U := (x_i y_i)_U \qquad (x_i)_{U*} := (x_i{}^*)_U.$$

For (iii) note that C(K)-spaces are C*-algebras and hence ultraproducts of C(K)-spaces are C(K)-spaces by the Gel'fand-Naimark theorem; (iv) requires the representation theorem for L^p spaces.

Finally, if $(E, \|\ \|, \phi)$ is a JB* triple system (cf. [2]), then there exists $M > 0$ such that for all $x,y,z \in E$

$$\|\phi(x,y)(z)\| \le M\|x\|\ \|y\|\ \|z\| \qquad (**)$$

so that $(\ell_\infty((E)_I), \|\ \|, \phi)$ is a JB* triple system with

$$\phi((x_i)_{i\in I}, (y_i)_{i\in I}) := (\phi(x_i,y_i))_{i\in I}.$$

N_U is a J* ideal in $\ell_\infty((E)_I)$ by (**) and hence $(E^I/U, \|\ \|, \phi)$ is a JB* triple system.

3. ULTRAPOWER PRINCIPLES AND SUPER-PROPERTIES

One of the successful typical applications of Banach space ultraproducts is in the local theory of Banach spaces, i.e. the study of the finite dimensional structure of Banach spaces and its relation to global structure. In particular, *finite representability* - the most important concept of the local theory - has a simple powerful ultrapower characterisation.

Let E and F be Banach spaces. F is *finitely representable* in E iff

for all $\varepsilon > 0$, for every finite dimensional subspace M of F, there exists a finite dimensional subspace N of E with $\dim N = \dim M$, and an isomorphism ϕ from M onto N such that

$$(1-\varepsilon)||x|| \leq ||\phi(x)|| \leq (1+\varepsilon)||x|| \quad \text{for all } x \in M.$$

The isomorphism ϕ is termed a $(1+\varepsilon)$ isomorphism. For orientation here are two results.

PROPOSITION 3.1.

(i) *Every Banach space is finitely representable in itself.*

(ii) *Finite representability is transitive.*

(iii) *Every Banach space is finitely representative in* ℓ_∞, *in* c_o, *and in the separable reflexive Banach space* $(\prod_{n\in N} \ell_n^\infty)_p$, *the* ℓ_p*-sum of the family* $\{\ell_n^\infty : n \in N\}$ *where* ℓ_n^∞ *is* C^n *with supremum norm* $(1 < p < \infty)$.

The easy proof is omitted. Incomparably deeper is:

THEOREM 3.2 (Dvoretzky). *ℓ_2 is finitely representable in every infinite dimensional Banach space.*

The advertised characterisation of finite representability is as follows:

THEOREM 3.3. *F is finitely representable in E iff there exists an ultrafilter U on a set I such that F is isometric to a subspace of E^I/U.*

PROOF. In the format of an expository note there is space just to isolate one characteristic feature of the proof of 3.3 which occurs in the choice of the index set I and the construction of the ultrafilter U on I.

Let I be the set of all pairs (M,ε) where M is a finite dimensional subspace of F and $\varepsilon > 0$. Partially order I by $<$: $(M_1,\varepsilon_1) < (M_2,\varepsilon_2)$ iff $M_1 \subset M_2$ and $\varepsilon_1 \geq \varepsilon_2$. Associate a filter A with $<$ on I:

$I_0 \in A$ iff $I_0 \subset I$ and there exists $(M_0,\varepsilon_0) \in I$ with

$$I = \{(M,\varepsilon) \in I : (M_0,\varepsilon_0) < (M,\varepsilon)\}.$$

Extend A to an ultrafilter U on I.

Since F is finitely representable in E, for each $i = (M_i,\varepsilon_i) \in I$, there exists a $(1+\varepsilon_i)$ isomorphism ϕ_i from M_i onto $N_i \subset E$:

$$(1-\varepsilon_i)||x|| \leq ||\phi_i(x)|| \leq (1+\varepsilon_i)||x|| \quad \text{for all } x \in M_i.$$

Define a mapping $J : F \to E^I/U$

$$Jx := (x_i)_U, \quad x_i = \begin{cases} \phi_i(x) & \text{if } x \in M_i, \\ 0 & \text{otherwise} \end{cases}$$

J is the required linear isometry.

Note in particular that E^I/U is finitely representable in E for any ultrafilter U on a non-empty set I.

3.2 and 3.3 imply that the modulus of convexity of a uniformly convex infinite dimensional Banach space is dominated by the modulus of convexity of ℓ_2.

Ultrapower techniques allow one to deduce information on the global structure of E from its local structure. The reformulation of local principles results in corresponding ultrapower principles. One of the best examples of this process is:

THEOREM 3.4 (Ultrapower Principle of local Reflexivity)

Let E be a Banach space. There exist an ultrafilter U on a set I and a mapping J from E^{**} into E^I/U such that

(i) J *is an isometric embedding of* E^{**} *into* E^I/U.

(ii) $J|_E$ *is the canonical embedding* $\bar{i}$ *of* E *into* E^I/U.

(iii) $J(E^{**})$ *is a norm-1 complemented subspace of* E^I/U, *i.e. there is a projection* P *of norm 1 onto* $J(E^{**})$.

PROOF. The Ultrapower Principle of Local Reflexivity is derived from the Principle of Local Reflexivity: (**) for all finite dimensional subspaces $M \subset E^{**}$, $N \subset E^*$ and $\varepsilon > 0$, there is a $(1+\varepsilon)$ isomorphism ϕ from M into E such that

(1) $\phi|_{M \cap E} = Id|_{M \cap E}$.

(2) $\langle f, \phi(x)\rangle = \langle x,f\rangle$ for all $x \in M$, $f \in N$.

Now proceed as in 3.3, taking I to be the set of all triples (M,N,ε) partially ordered with an ultrafilter U on I. Use (**) to define a mapping $J : E \to E^I/U$

$$Ju := (x_i)_U, \quad x_i = \begin{cases} \phi_i(x) & \text{if } x \in M_i, \\ 0 & \text{otherwise.} \end{cases}$$

Parts (i) and (ii) follow.

To complete the proof, define a mapping $Q : E^I/U \to E^{**}$
$Q((x_i)_U) := \lim_U x_i$.
Note that by 1.1 and the weak * compactness of the closed balls of E^{**} the limit is well-defined (identifying E with its canonical image in E^{**}). Set $P := J \circ Q$ to obtain the required projection of norm 1 onto $J(E^{**})$.

One corollary of 3.4 is this: if B is a class of Banach spaces which is closed under ultraproducts and contractive projections (P is a contractive projection iff $P^2 = P$ and $||P|| \le 1$ then B is closed under formation of biduals, i.e.

$$E \in B \text{ implies } E^{**} \in B.$$

The class of JB* triple systems is closed under contractive

projections, so from 2.1 one deduces the following recent theorem of S. Dineen [2]:

THEOREM 3.5. *Let* $(E, || \ ||, \phi)$ *be a JB* triple system. Then the bidual* $(E^{**}, || \ ||, \phi)$ *is a JB* triple system.*

Intuitively, a *local property* of Banach spaces is a property P such that if E has P, then every Banach space locally similar to E also has P. Super-properties are the mathematically precise explication of this intuition. Let P be any property of Banach spaces. E has the property *super*-P iff every Banach space finitely representable in E has the property P. P is called a *super-property* iff whenever E has P then E has super-P.

Examples of super-properties include: uniform convexity, super-reflexivity, the properties "E is finitely representable in G" and "G is not finitely representable in " for arbitrary fixed Banach space G.

The super-properties of infinite dimensional Banach spaces can be ordered in a hierarchy: there is a weakest (trivial) super-property W, the first (non-trivial) super-property C, and the strongest super-property H. Their definitions run:

H(E) : E is a Hilbert space.

C(E) : c_0 is not finitely representable in E.

W(E) : E is infinite dimensional.

3.1 and 3.2 show that the following implications hold:
$H(E) \Rightarrow Q(E) \Rightarrow C(E) \Rightarrow W(E)$ where Q is any super-property. 3.2 implies too that W is equivalent to the super-property D;

D(E) : ℓ_2 is finitely representable in E.

There are many characterisations of C. Here is a recent one deriving from results in [3]. Let BD be the property:

BD(E) : every bounded domain in the complex Banach space E is biholomorphically equivalent to a finite product of irreducible complex Banach manifolds.

Then C is equivalent to super-BD.

Immediate consequences of the hierarchy of super-properties are:

(1) Hilbert spaces possess every super-property;

(2) if ℓ_2 fails to have a given super-property Q, then no infinite dimensional Banach space has Q;

(3) if E is infinite dimensional with even one (non-trivial) super-property, then c_0 is not finitely representable in E.

4. CONCLUSION

The Banach space ultraproduct was developed initially in an interaction of functional analysis and mathematical logic. Thus it is not surprising to find Banach space analogues of theorems of first-order model theory: downward Loewenheim-Skolem theorem, Keisler-Shelah theorem ([8], [1]). A simple corollary of these results is a version of the Banach-Mazur theorem:

<u>COROLLARY 4.1</u>. *Assuming the continuum hypothesis, there exists a Banach space of density character $\aleph_1$ which contains (isometrically isomorphic copies of) every Banach space of density character at most $\aleph_1$. In fact, there is an ultrapower of c_0 satisfying 4.1.*

Recent applications of ultraproduct techniques to non-linear classification problems can be found in [7].

Finally, the material of Section 2 can be generalized to define ultrapowers of locally convex spaces [5], [6].

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