EXTENSIONS AND K-THEORY OF C*-ALGEBRAS

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INTRODUCTION

The theory of C*-algebras is increasingly having an impact on other areas of Mathematics, and on Mathematical Physics, for example, on Algebraic Topology, Differential Geometry, Topological Group Theory and Quantum Mechanics. Our aim here is to give an account, comprehensible to the non-specialist, of some of the most important recent results in this subject.

THE BROWN-DOUGLAS-FILLMORE THEORY

Let H be a Hilbert space (all vector spaces and algebras are over the complex number field C). An operator T on H is *normal* if T*T = TT* and such an operator is *diagonalizable* if H admits an orthonormal basis consisting of eigenvectors of T. Of course relative to such a basis T has diagonal matrix, and on finite dimensional Hilbert spaces all normal operators are diagonalizable, but this is false in infinite dimensions: if $(e_n)_{n \in Z}$ is an orthonormal basis for H and if T in B(H) (the algebra of all bounded linear operators on H) is defined by $Te_n = e_{n+1}$ ($n \in Z$) then T is normal and a trivial calculation shows that T has no eigenvectors.

However despite this negative result, in a certain sense normal operators are "nearly diagonalizable". To be precise, H. Weyl (1909) showed that if H is a *Hermitian* operator (T = T* on a separable infinite-dimensional Hilbert space H then T is a sum of a diagonalizable operator and a compact operator (an operator K on H is *compact* if there is a sequence of operators K_n with finite dimensional ranges such that $||K_n - K||$ converges to 0 as n tends to ∞, where $||.||$ denotes the operator norm on B(H). From the point of view of Operator Theory compact operators are "small", and adding on a compact operator to a given operator only "perturbs" the operator "inessentially").

By the way, the Weyl result fails if separability is dropped. Surprisingly, the extension of this result to all normal operators did not come until 1970 when I.D. Berg showed that every normal operator on a separable infinite dimensional Hilbert space is the sum of a diagonalizable and a compact operator.

Now let us consider the set *D+K* of all sums D+K where D is a diagonalizable and K a compact operator on H (*henceforth H will always denote a separable infinite-dimensional Hilbert space*). Given an operator T on H one could ask for a "spectral condition" on T that T belong to *D+K*. This is not very precise, but if T ∈ *D+K* then its self-commutator T*T-TT* is compact, i.e. T is *essentially normal*. One could now ask (naively) do all essentially normal operators belong to *D+K*? The answer is no, and the explanation is elementary but revealing.

An operator S on H is *Fredholm* if it has closed range and the spaces N(S) and N(S*) are finite dimensional (N( ) denotes the null-space or kernel). We then define the *Fredholm index* of S to be

$$\text{index}(S) = \text{dimension } N(S) - \text{dimension } N(S^*).$$

One has index(S) = index(S+K) for all compact operators K, and if S is normal, index(S) = 0. Now let $(e_n)_{n \in N}$ be an orthonormal basis for H and let U be the operator on H defined by $Ue_n = e_{n+1}$ ($n \in N$). U is called the *unilateral shift* and will be referred to again later. U is essentially normal and of Fredholm index -1. Thus U cannot be of the form diagonal + compact, since any such operator is of index 0.

It turned out that this index obstruction was the *only* obstruction, but the proof of this required the introduction of homological algebra techniques into Operator Theory. First we shall state the results of the beautiful theory of L. Brown, R. Douglas and P. Fillmore (1973) and then we shall indicate briefly their approach to the problem.

If $S \in B(H)$ its *essential spectrum* is the set $\sigma_e(S)$ of all complex numbers λ such that $S-\lambda 1_H$ is not a Fredholm operator.)

THEOREM (B-D-F, 1973)

1. *If T is an essentially normal operator on H then T is a sum of a diagonalizable operator and a compact operator if and only if* $\text{index}(T-\lambda 1_H) = 0$ *for all* $\lambda \in C \setminus \sigma_e(T)$.

2. *If* T_1, T_2 *are essentially normal operators on H then ther[e] is a compact operator K on H such that* T_2*-K is unitarily equiv[alent] to* T_1 *(in this case we say* T_1 *and* T_2 *are compalent if an[d] only if* T_1 *and* T_2 *have the same essential spectrum X and for all* $\lambda \in C \setminus X$ *we have* $\text{index}(T_1-\lambda 1_H)$).

Thus this theorem completely classifies the essentially normal operators (a very large and important class of operator[s]) up to unitary equivalence modulo the compact operators. Alth-ough the results are stated in simple Operator Theoretic terms the proofs involve algebras of operators, i.e. C*-algebras, an[d] as mentioned above, homological algebra. Many Operator Theor-ists would prefer proofs that did not involve the latter, and seems that at last this may be possible, for only this year (1986) I.D. Berg and K. Davidson have announced a new proof o[f] the B-D-F theorem that apparently uses quite different method[s]. However, homological methods are here to stay in Operator Alg[ebra] Theory, since there now exist many more deep results usi[ng] these methods, some of which we'll be looking at later.

THE THEORY OF EXTENSIONS

A *C*-algebra* is a Banach algebra A with an isometric in-volution $x \mapsto x^*$ such that $||x^*x|| = ||x||^2$ for all x in A. I[f] X is a compact Hausdorff space then C(X), the set of all comp[lex-]valued continuous functions on X, is a C*-algebra with the ob[v]ious pointwise-defined operations and the supremum norm. If [K] is any Hilbert space, then B(K) is a C*-algebra (the norm is the operator norm, and the involution is defined by the

usual adjoint operation). All self-adjoint closed subalgebras of B(K) are C*-algebras and the Gelfand-Naimark theorem says that every C*-algebra has a faithful representation as such a C*-algebra.

If A and I are C*-algebras then an *extension* of A by I is a short exact sequence of C*-algebras and *-homomorphisms

$$0 \to I \to E \to A \to 0.$$

(If A,B are C*-algebras a **-homomorphism* from A to B is an alg-ebra homomorphism $\alpha: A \to B$ which preserves the involution, $\alpha(x^*) = (\alpha(x))^*$ for all x in A. We say α is *unital* if A and B have multiplicative identity elements 1_A and 1_B and $\alpha(1_A) = 1_B$.)

Our definition of extension is too general for the present purpose, since we shall only be interested in extensions of C(X), for X a compact Hausdorff space, by K(H), the C*-algebra of all compact operators on H. Thinking of extensions as short exact sequences is a little clumsy, so we shall present them in an equivalent but more convenient form.

Henceforth X denotes a compact metrizable space.

An *extension* of C(X) (by K(H)) will mean an injective unital *-homomorphism $\tau: C(X) \to B(H) / K(H)$ (this quotient alg-ebra is a C*-algebra with the quotient norm and obvious invol-ution: it is called the *Calkin* algebra). We say two extensions τ_1, τ_2 of C(X) are *equivalent* if there exists a unitary oper-ator U in B(H) (i.e. $U^*U = UU^* = 1$) such that $\tau_2(f) = \pi(U)\tau_1(f)\pi(U^*)$ for all f in C(X). Here π denotes the quotient map from B(H) to B(H) / K(H). This defines an equivalence relation and we denote the class of τ by $[\tau]$, and the set of these equiv-alence classes by Ext(X). We'll see shortly that Ext(X) can be made into a group.

Now let $T \in B(H)$ be essentially normal. Then $\pi(T)$ is a normal element of the Calkin algebra, i.e. $\pi(T)$ and $\pi(T)^*$ commute, so by the Spectral Theorem there exists a unique unital injective *-homomorphism τ_T from $C(\sigma_e(T))$ to the Calkin algebra such that $\tau_T(z) = \pi(T)$, where z denotes the inclusion map of $\sigma_e(T)$ in C. We call τ_T the *extension of* $\sigma_e(T)$ *determined by* T. If two essentially normal operators on H both have essential spectrum X then they are compalent iff the extensions they determine are equivalent.

Given a general compact metrizable space, extensions of C(X) exist. In fact trivial extensions exist, where the extension τ is said to be *trivial* if there is a unital *-homomorphism ρ from C(X) to B(H) such that $\tau = \pi\rho$. The trivial extensions form the zero of Ext(X). (By the way if metrizablity of X is dropped then trivial extensions may not exist.)

The first important result of this theory is that all trivial extensions of C(X) are equivalent. The proof uses Weyl's theorem, and Berg's theorem drops out as a consequence of this result. An addition can be defined on Ext(X) in a natural way (using direct sums of operators), and one can show easily that Ext(x) is a commutative semigroup. The fact that the class of the trivial extensions forms the zero of Ext(X) is a non-trivial result - using it and the Wold-von Neumann decomposition of isometries (an *isometry* is an operator U in B(H) such that U*U = 1) one can show the following:

<u>THEOREM (B-D-F, 1973)</u>. *Let* $U \in B(H)$ *be the unilateral shift and let* $T \in B(H)$ *be an essentially unitary operator (i.e.* $\pi(T)^*$ *is the inverse of* $\pi(T)$ *in the Calkin algebra, so in particular T is essentially normal), and let n be the Fredholm index of T. Then there exists* $K \in B(H)$ *compact such that*

1. $T-K$ *is unitary if* $n = 0$.
2. $T-K = U^n$ *if n is negative.*
3. $T-K = U^{*n}$ *if n is positive.*

It follows that Ext(T) = Z where T is the unit circle.

As mentioned earlier Ext(X) is a group. The original B-D-F proof of this was very complicated but W. Arveson has simplified the proof.

Our next task is to "identify" the group Ext(X), at least for X a compact subset of C (the case relevant to single operators).

Let $\pi^1(X)$ denote the first cohomotopy group of X (this can be identified as the quotient group of the group of invertible elements of C(X) modulo the connected component of 1 (which is a subgroup). Equivalently $\pi^1(X)$ is the group of homotopy classes of continuous functions from X to $C \setminus \{0\}$). $Hom(\pi^1(X), Z)$ denotes the group of all homomorphisms from $\pi^1(X)$ to Z, and we define a map γ_X from Ext(X) to this group by the equation

$$\gamma_X[\tau][f] = \text{index}(\tau(f))$$

where $[\tau] \in Ext(X)$, f is an invertible element of C(X), and [f] denotes the class of f in $\pi^1(X)$. $\gamma = \gamma_X$ is easily seen to be a homomorphism, but it is not an isomorphism is general. However it is a deep result of the B-D-F theory that γ is an isomorphism if X is a compact subset of the plane. It is here that homological algebra comes in, and surprisingly perhaps, one has to be able to talk about Ext(X) for X not a subset of the plane to construct the proof.

We are now ready to sketch a proof of the B-D-F theorem: let T_1, T_2 be essentially normal operators on H with essential spectrum X and suppose that $\text{index}(T_1-\lambda 1) = \text{index}(T_2-\lambda 1)$ for all $\lambda \in C \setminus X$. We have to show that the extensions τ_1 and τ_2 determined by T_1 and T_2 respectively are equivalent extensions, and to do this it suffices to show that $\gamma_X[\tau_1] = \gamma_X[\tau_2]$. But $\pi^1(X)$ is generated by the elements $[z-\lambda_\omega]$ where z is the inclusion map of X in C and λ_ω is an arbitrary point of the hole ω (a *hole* of X is a bounded connected component of $C \setminus X$). Thus it suffices to show that $\gamma_X[\tau_1][z-\lambda_\omega] = \gamma_X[\tau_2][z-\lambda_\omega]$, i.e.

$\text{index}(\tau_1(z-\lambda_\omega)) = \text{index}(\tau_2(z-\lambda_\omega))$, i.e. $\text{index}(T_1-\lambda_\omega) = \text{index}(T_2-\lambda_\omega)$. But this is true since $\lambda_\omega \in C \setminus X$.

Brown, Douglas and Fillmore went on to show that one can use Ext to define a generalized periodic homology theory on compact metric spaces. Their theory thus links up with the theory of pseudodifferential operators and the Atiyah-Singer index theorem. We are not going to pursue this; instead we look at a theory which is in a sense dual to that of extensions, which has already had many important applications, and which is in some respects more "natural" than the theory of extensions.

K-THEORY OF C*-ALGEBRAS

The basic idea of K-theory is that we can analyse a C*-algebra in terms of the projections and unitaries that it - or rather the matrix algebras $M_n(A)$ - contain. To avoid technical difficulties we assume that A is unital. A *projection* in A is a self-adjoint idempotent element $p : p = p^2 = p^*$. A *unitary* u in A is an element whose adjoint is its inverse. We let $M_\infty(A)$ denote the set of infinite matrices with entries in A and with only finitely many entries non-zero. With the obvious matrix operations this is an involutive normed algebra (each subalgebra $M_n(A)$ has a unique norm making it a C*-algebra, and $M_\infty(A)$ is the union of all $M_n(A)$ $(n = 1,2, \dots)$).

We say projections e,f in A are *equivalent* if there is a continuous path of projections in $M_\infty(A)$ from e to f. H(A) denotes the set of equivalence classes [e] for this equivalence relation. If $s,t \in H(A)$ then there exist projections e,f in $m_\infty(A)$ such that $ef = 0$ and $s = [e]$, $t = [f]$. We define (without ambiguity) $s+t = [e+f]$. This makes H(A) an abelian semigroup with zero element, and we let $K_0(A)$ denote its Grothendieck enveloping group (loosely speaking the set of all formal differences $[e] - [f]$). The definition of $K_0(A)$ for A non-unital is got from the unital case. The details

are unenlightening and straightforward, so omitted.

Now for the definition of K_1 - this simpler than K_0 and we do not have to assume that A is unital. We let $M_\infty(A)^\sim$ be the involutive normed algebra got by adjoining an identity element to $M_\infty(A)$. U(A) denotes the group of unitaries of $M_\infty(A)^\sim$ and U_1 the connected component of 1 in U. This is a normal subgroup of U and we let $K_1(A)$ denote the quotient group U/U_1.

One should think of $K_0(A)$ as an "index" group - many Fredholm-type indices have their values in some $K_0(A)$. Specifically one should think of [e] as the "dimension" of the projection e. In some ways K-theory is like a generalized Fredholm index theory.

Here are a few random examples of K-groups:

1. $K_0(C) = Z$, $K_1(C) = 0$.

2. $K_0(B(H)) = 0$, $K_1(B(H)) = 0$.

3. If O_n is the C*-subalgebra of B(H) generated by n $(n > 1)$ isometries $S_1, \dots, S_n$ such that $S_1S_1^* + \dots + S_nS_n^* = 1$ then O_n is simple (i.e. it has no proper closed two-sided ideals) and $K_0(O_n) = Z/(n-1)$. Thus the K-groups can have torsion.

Before listing the basic properties of K-theory we look at some of the applications of the theory.

An *AF-algebra* is a C*-algebra having an increasing sequence of finite-dimensional C*-subalgebras $A_1 \subset A_2 \subset \dots$ such that the union $\cup(A_n : n = 1,2, \dots)$ is dense in A. Some examples are c_0, K(H), and the CAR-algebra so important to mathematical physicists. This class of algebras is diverse and extensive - for example there are uncountably many non-isomorphic simple AF-algebras. The AF-algebras exhibit typical C*-algebra behaviour and are highly non-trivial in general - they are usually not even type I.

If A is an AF-algebra and I is a closed two-sided ideal in A then I and A/I are AF-algebras. It is natural to enquire if the converse is true, i.e. if A is a C*-algebra and I a closed two-sided ideal in A such that both I and A/I are AF-algebras is A an AF-algebra? L. Brown answered this affirmatively in one of the first applications of K-theory. The essential idea was to show that one can lift projections from the quotient algebra A/I to A, i.e. to show that every projection in A/I is the image of a projection in A under the quotient map π from A to A/I. The proof used the 6-term exact sequence of K-theory (this sequence will be exhibited below).

Although not originally conceived in K-theoretic terms it was soon realized that the classification of AF-algebras due to O. Bratteli and G. Elliott (1978) involved K_0 ($K_1(A) = 0$ for any AF-algebra A). For simplicity we'll restrict ourselves to unital AF-algebras. One can define a translation-invariant partial ordering on $K_0(A)$ (for A any unital AF-algebra) by defining the positive cone $K_0(A)^+$ to be the set of all [e] for e a projection in $M_\infty(A)$. Thus $K_0(A)$ becomes a partially ordered group. Now the C*-algebras A and B are said to be *stably isomorphic* if the C*-tensor product of K(H) and A is *-isomorphic to the C*-tensor product of K(H) and B - loosely speaking this means that A and B have the same representation theory. Stably isomorphic C*-algebras have the same K-groups. One of the elegant results of Bratteli and Elliott is that unital AF-algebras A, B whose K_0-groups are isomorphic as partially ordered groups, are stably isomorphic. Moreover if there exists a partially odered group isomorphism ϕ from $K_0(A)$ to $K_0(B)$ such that $\phi[1_A] = [1_B]$ then A and B are actually *-isomorphic!

One can use $K_0(A)$ to investigate the structure of the AF-algebra A. For example, there is a bijective correspondence

between the lattice of closed two-sided ideals of A and the lattice of "ideals" (certain subgroups) of $K_0(A)$. Also, one can give an abstract characterization of the partially ordered groups that can appear as $K_0(A)$ for some AF-algebra A (Effros-Handelmann-Shen 1980). These groups are called *dimension groups*. Let us mention in passing an application of this to Quantum Mechanics: one can use this theory to construct C*-dynamical systems with a given set of temperatures for KMS states. However one of the most striking applications of the theory was its use in solving a long-standing open problem posed by I. Kaplansky in 1958, namely is there a simple C*-algebra, other than C, with no non-scalar projections? B. Blackadar (1980) constructed a certain dimension group having an unusual automorphism property, and this was reflected in an unusual automorphism property of the corresponding AF-algebra. He then used this AF-algebra to construct a simple C*-algebra with no non-trivial projections. Nevertheless this still left open a conjecture of R. Kadison that a certain C*-algebra $C^*_{red}(F_2)$ (a stock counterexample in many situations) had no non-trivial projections. This question was resolved affirmatively by M. Pimsner and D. Voiculescu using K-theory. A Connes gave an alternative proof again using K-theory, but also using his "non-commutative Differential Geometry" (another new exciting area in C*-algebra theory). By the way K-theory has been successfully applied in classical Differential Geometry, to the Novikov Conjecture on the homotopy invariance of higher signatures.

Now it is time to list some of the basic properties of K-theory:

1. If $\alpha : A \to B$ is a *-homomorphism of C*-algebras then there are corresponding group homomorphisms $K_j(\alpha) : K_j(A) \to K_j(B)$ for $j = 0,1$. This defines a pair of covariant functors from the category of all C*-algebras to the category of all abelian groups.

2. (Continuity) If A is an inductive limit in the category of C*-algebras, $A = \lim_\lambda A_\lambda$ say, then $K_j(A) = \lim_\lambda K_j(A_\lambda)$, $j=0,1$.

3. (Homotopy Invariance) If ϕ_t $(0 \le t \le 1)$ is a continuous path of *-homomorphisms from the C*-algebra A to the C*-algebra B (in the topology of pointwise convergence) then $K_j(\phi_0) = K_j(\phi_1)$ j = 0,1.

4. Stably isomorphic C*-algebras have isomorphic K-groups.

5. (Bott Periodicity) Let B be the C*-tensor product of the C*-algebra A and $C_0(R)$. Then $K_1(A) = K_0(B)$ and $K_0(A) = K_1(B)$.

6. (Periodic Exact Sequence) If I is a closed two-sided ideal in the C*-algebra A then there is an exact sequence

$$K_0(I) \to K_0(A) \to K_0(A/I) \to K_1(I) \to K_1(A) \to K_1(A/I) \to K_0(I)$$

The boundary map δ_1 from $K_1(A/I)$ to $K_0(I)$ can be thought of as a sort of generalized Fredholm index.

5 and 6 are deep and powerful theorems.

A FEW CONCLUDING REMARKS

The theory of extensions and K-theory have been synthesized into a new theory, KK-theory. Readable accounts of K-theory and extensions are to be found in [1], [2] and [3]. Extensive bibliographies are to be found in [1] and [3].

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DIFFERENTIAL EQUATIONS

NIHE DUBLIN

A meeting on *Differential Equations* will take place between 27th - 29th May at the National Institute for Higher Education, Dublin. There will be special sessions on nonlinear equations and on asymptotics for linear equations. Invited speakers include K. Brown (Heriot-Watt), M.S.P. Eastham (King's College, London), H. Ockendon (Oxford), J.R. Ockendon (Oxford) and R.B. Paris (CEA - Euratom). Presentations on any aspect of differential equations (pure or applied) are welcome. Further information can be obtained from Prof. A. Wood, School of Mathematical Sciences, National Institute for Higher Education, Dublin, Dublin 9, Republic of Ireland.