# Problem Solving
# Answers to Problem Set 9

12 July 2012

1. Let $x, y$, and $z$ be integers such that $n = x^4 + y^4 + z^4$ is divisible by 29. Show that $n$ is divisible by $29^4$.

   **Answer:** *Hopefully we shall find that $x, y, z$ are all divisible by 29. (In fact, this must be the case if the result is true. For suppose there was a solution $x, y, z$ with $x$, say, not divisible by 29. Then $x + 29, y, z$ would satisfy the first condition, but not the second since $(x + 29)^4 - x^4$ is not divisible by $29^4$.*

   *Consider the multiplicative group $(\mathbb{Z}/29)^*$ formed by the non=zero residues $1, 2, \ldots, 28 \bmod 29$. We know that this group is cyclic (the Primitive Root Theorem), isomorphic to $C_{28}$. It follows that the 4th powers modulo 29 form a cyclic group of order 7. (If $\pi$ is a primitive root modulo 29 then these are the elements $\pi^{4i}$ for $0 \leq i < 7$.*

   *This means there are only 7 4th powers we need consider. We have*

$$\begin{aligned}
1^4 &\equiv 1 \bmod 29,\\
2^4 &\equiv 16 \equiv -13 \bmod 29,\\
3^4 &\equiv 81 \equiv -6 \bmod 29,\\
4^4 &\equiv (2^4)^2 \equiv 13^2 = 169 \equiv 24 \equiv -5 \bmod 29,\\
6^4 &\equiv 2^4 \cdot 3^4 \equiv 13 \cdot 6 = 78 \equiv 20 \equiv -9 \bmod 29,\\
8^4 &\equiv (4^4)^2 \equiv 25 \equiv -4 \bmod 29,\\
9^4 &\equiv (3^4)^2 \equiv 6^2 \equiv 7.
\end{aligned}$$

*So the 4th powers are* $1, 7, -4, -5, -6, -9, -13$.

*It is clear that no three of these numbers can add up to* $+29$ *or* $-29$. *So if they add to* $0 \mod 29$ *they will have to actually add to 0. So there will have to be one or two positive numbers. There cannot be two positive numbers, since* $-2, -8, -14$ *are not in the list. So there must be one positive number and two negative numbers. But again, we cannot get* $-1$ *or* $-7$ *as a sum of two of the negative numbers.*

*Also, we cannot have just one of* $x, y, z$ *divisible by* $29$ *since the remaining two numbers would have to be* $\equiv \pm n \mod 29$, *and there is no such possibility.*

*We conclude that* $x, y, z$ *are all divisible by 29, and the result follows.*

2. Find all continuous odd functions $f : \mathbb{R} \to \mathbb{R}$ such that the identity $f(f(x)) = x$ holds for all real $x$.

   **Answer:** *The identity* $I(x) = x$ *and* $J(x) = -x$ *are two such functions. But are they the only ones?*

   *Evidently*

$$f(0) = 0$$

   *since*

$$f(0) = f(-0) = -f(0).$$

   *Also* $f$ *is bijective, being its own inverse. Since it is continuous, it is a homeomorphism. It follows that the two components of* $\mathbb{R} \setminus \{0\}$, *namely*

$$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\} \text{ and } \mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$$

   *must map into one another.*

   *If* $f$ *swaps the two components, then* $Jf$ *(which must also be a solution to the problem) maps each component into itself. We may assume therefore that this is true of* $f$.

   *A homeomorphism of* $\mathbb{R}$ *is necessarily strictly monotone. In this case since* $f(0) = 0$ *it must be monotone increasing.*

*Suppose $f$ is not the identity, say*

$$f(x) = y > x.$$

*Applying $f$,*

$$f(f(x)) = x > f(x),$$

*which is a contradiction. Similarly if $f(x) < x$.*

*Hence $f = I$, and $I, J$ are the only functions satisfying the conditions in the question.*