Problem Solving
Answers to Problem Set 6

09 July 2012

1. Let f be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer k. Prove that all coefficients of f are divisible by 5.

   Answer: *This is really very easy; it is just a matter of working modulo 5.*

   Suppose
   $$f(x) = ax^2 + bx + c.$$
   Then $f(0) = c$, so
   $$5 \mid f(0) \implies 5 \mid c.$$
   So we may assume that $c = 0$, by passing to $f(x) - c$.

   We have $f(1) = a + b$ and
   $$f(2) = 4a + 2b \equiv -a + 2b \bmod 5.$$
   Adding,
   $$3b \equiv 0 \bmod 5 \implies b \equiv 0 \bmod 5.$$
   Finally,
   $$a + b \equiv 0 \bmod 5 \text{ and } b \equiv 0 \bmod 5 \implies a \equiv 0 \bmod 5.$$

2. (a) Given a triangle ABC show that there is a unique triangle PQR with the points P, Q, R on the sides BC, CA, AB, and the edges QR, RP, PQ parallel to BC, CA, AB, respectively.

(b) Suppose we inscribe a third triangle in the same way inside PQR, and a fourth triangle inside this one, and so on. Show that the areas of the triangles form a geometric sequence.

Answer:

(a) Take P, Q, R to be the mid-points of BC, CA, AB. Then the triangles AQR and ACB are similar (two pairs of sides proportional, and included angles equal). Hence the angles $A\hat{Q}R$ and $A\hat{C}B$ are equal, and so QR is parallel to CB; and the same is true for the other sides.

To see that this solution is unique, if PQR is inscribed as stated then the triangle PQR is congruent to each of the triangles AQR, PBR, PQC. It follows that $BP = CP$, so P is the mid-point of BC; and similarly for Q and R.

(b) Since ABC is divided into 4 congruent triangles, the area of PQR is $1/4$ the area of ABC. Similarly the area of the next inscribed triangle is $1/4$ the area of PQR or $1/4^2$ the area of ABC, and so. The areas form a geometric series, with ratio $1/4$.