Problem Solving
Answers to Problem Set 5

08 July 2012

1. Let f be a real-valued function with $n+1$ derivatives at each point of $\mathbb{R}$. Show that for each pair of real numbers $a, b, a < b$, such that

$$\ln\left(\frac{f(b)+f'(b)+\cdots+f^{(n}(b)}{f(a)+f'(a)+\cdots+f^{(n}(a)}\right)=b-a$$

there is a number c in the open interval (a, b) for which

$$f^{(n+1)}(c)=f(c).$$

Answer: *Let*

$$F(x)=f(x)+f'(x)+\cdots+f^{(n)}(x).$$

Then

$$F'(x)-F(x)=f^{(n+1)}(x)-f(x).$$

We are told that

$$\frac{\ln F(b)-\ln F(a)}{b-a}=1.$$

By the Mean Value Theorem,

$$\frac{\ln F(b)-\ln F(a)}{b-a}=\frac{F'(c)}{F(c)}$$

for some $c \in (a, b)$.

Thus

$$\frac{F'(c)}{F(c)} = 1,$$

ie

$$F'(c) = F(c),$$

ie

$$f^{(n+1)}(c) - f(c).$$

2. Given one million non-zero digits $a_1, a_2, \ldots, a_{1000000}$ (ie each $a_i \in \{1, 2, \ldots, 9\}$) show that at most 100 of the million numbers

$$a_1,\ a_1a_2,\ a_1a_2a_3,\ \ldots,\ a_1a_2a_3\ldots a_{1000000}$$

are perfect squares

Answer: *This is based on an idea that comes up occasionally — squares are quite widely distributed, since there is no square between n^2 and $n^2 + n$,*

Consider the numbers in the subsequence with an even number of digits, omitting the first, ie

$$a_1a_2a_3a_4,\ a_1a_2a_3a_4a_5a_6,\ \ldots.$$

We claim that there is at most one square in this sequence. For suppose the first square in the sequence is

$$a_1a_2\ldots a_{2r} = b^2;$$

and suppose there is a later square in the sequence

$$a_1a_2\ldots a_{2r}\ldots a_{2r+2s} = c^2.$$

Then

$$(10^s b)^2 = 10^{2s} b^2 = a_1a_2\ldots a_{2r}\overbrace{0\ldots0}^{2s\ 0\text{'s}}$$

Now

$$(10^s b+1)^2 = 10^{2s}b^2 + 2b\cdot 10^s + 1$$
$$> a_1a_2\ldots(a_{2r}+1)\overbrace{0\ldots0}^{2s\ \mathit{0's}},$$

which is not in the sequence.

Similarly there is at most one square in the subsequence with an odd number of digits, omitting the first, ie

$$a_1a_2a_3,\ a_1a_2a_3a_4a_5,\ \ldots.$$

It follows that there are at most $2+e$ *squares in the sequence, where* e *is the number of squares with 1 or 2 digits. Evidently* $e=9$*; so there are at most 11 squares in the sequence.*