Problem Solving
Answers to Problem Set 4

07 July 2012

1. Let $f \in C^1[a,b],\ f(a) = 0$ and suppose that $\lambda \in \mathbb{R},\ \lambda > 0$ is such that

$$|f'(x)| \le \lambda\,|f(x)|$$

for all $x \in [a,b]$. Is it true that $f(x) = 0$ for all $x \in [a,b]$?

Answer: *My first thought is that this says $f(x)$ is 'sub-exponential' in some sense; and if $|f(x)| < Ce^{\lambda x}$ then $f(a) = 0$ will imply that $f(x) = 0$. Can we make this rigorous?*

I think one could do it by integrating $f'(x)/f(x)$. But it is probably easier to use the Mean Value Theorem.

First, assuming the result is false we can replace a by the largest $t \in [a,b]$ such that $f(x) = 0$ on $[a,t]$.

So we may assume that there are reals $x \in [a,b]$ arbitrarily close to a such that $f(x) \neq 0$.

Choose $c \in [a,b]$ close to a. (We will decide later what this means.) Let

$$d = max_{x\in[a,c]}\,|f(x)|\,.$$

By the Mean Value Theorem

$$\begin{aligned} f(d) &= f(d) - f(a) \\ &= (d-a)f'(t) \end{aligned}$$

for some $t \in (a,d)$.

But then, by the hypothesis in the question

$$\begin{aligned}|f(d)| &\le (d-a)\lambda\,|f(t)| \\ &\le (d-a)\lambda\,|f(d)|\,.\end{aligned}$$

It follows that

$$d - a \ge 1/\lambda,$$

which cannot hold if we choose

$$c - a < 1/\lambda.$$

2. What is the greatest sum that cannot be paid for in 2c and 5c coins?

   Answer: *This is a simple exercise in the Chinese Remainder Theorem.*

   We know that given n we can solve

   $$2x + 5y = n$$

   in integers *(positive or negative).*

   Suppose $n \ge 10$; and suppose

   $$2x + 5y = 10,$$

   with $x, y \in \mathbb{Z}$. Then $(x_0, y_0) = (x + 5t, y - 2t)$ will also satisfy the equation

   $$2x_0 + 5y_0 = 10,$$

   for any $t \in \mathbb{Z}$.

   We can choose t such that $x_0 \in [0, 5)$. Then $2x_0 < 10$ and so $y_0 > 0$. Thus we have a solution of the equation with $x, y \ge 0$.

   So we only need to consider $0 \le n < 10$; and it is evident that the largest integer not *expressible in the form $2x + 5y$ with $x, y \ge 0$ is 3.*

By the same argument, if m, n are coprime then any integer $\geq mn$ is expressible in the form $mx + ny$ with $x, y \geq 0$.

It's a little more difficult to show that the largest integer not *expressible in this form is $mn - (m + n)$. This uses the uniqueness modulo mn) part of the Chinese Remainder Theorem.*