Problem Solving
Answers to Problem Set 1

04 July 2012

1. Let $f \in C^1(a, b)$, $\lim_{x \to a+} f(x) = +\infty$, $\lim_{x \to b-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a, b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

   Answer: *I misread this when I first read it (without my glasses), as I read f^1 for f', and so thought f^2 meant f''. Which is probably irrelevant, except to read the problems carefully, and make sure you are wearing glasses if necessary!*

   But once I had understood the problem, I wrote it as

   $$\frac{df}{f^2 + 1} \geq -dx.$$

   (It's probably not fashionable to treat df, dx as "real" quantities, but it can be justified, and I often find it helpful.)

   We may assume that $a = 0$, on replacing $f(x)$ by $f(x - a)$. We have to show that $b \geq \pi$.

   Integrating,

   $$\int_{\infty}^{-\infty} \frac{df}{f^2 + 1} \geq -b,$$

   ie

   $$[\arctan(f)]_{\infty}^{-\infty} \geq -b,$$

ie

$$-\pi \geq -b,$$

or

$$b \geq \pi.$$

2. A collection of subsets of $\{1, 2, \ldots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.

   Answer: *I got this from a collection of problems on the pigeon-hole principle, so that gave me a starting-point.*

   Let $N = \{1, 2, \ldots, n\}$, *and let* $S = 2^N$, *ie the set of* 2^n *subsets of* N. *We are told we have a subset*

$$X \subset S,$$

   ie a collection of subsets of N.

   To apply the pigeon-hole principle we must divide S *into* 2^{n-1} *subsets, ie with an average of 2 elements per subset.*

   The obvious way is to pair off the complementary subsets $\{Z, N \setminus Z\}$ *of* S. *The pigeon-hole principle tells us that if we have a subset* $X \subset S$ *containing* $> 2^{n-1}$ *elements then two of these elements must be in the same subset* $\{Z, N \setminus Z\}$.

   Since Z *and* $N \setminus Z$ *have no element in common, this will contradict the hypothesis of the question. Hence there must be* $\leq 2^{n-1}$ *subsets.*

   [I was told that this makes the question seem much more complicated than it really is!]