18^{th} IMC Competition

2012

A1. For every positive integer n, let $p(n)$ denote the number of ways to express n as a sum of positive integers. For instance, $p(4) = 5$ because $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$. Also define $p(0) = 1$. Prove that $p(n) - p(n-1)$ is the number of ways to express n as a sum of integers each of which is strictly greater than 1.

A2. Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

A3. Given an integer $n > 1$, let S_n be the group of permutations of the numbers $1, 2, \ldots, n$. Two players, A and B, play the following game. Taking turns, they select elements (one element at a time) from the group S_n. It is forbidden to select an element that has already been selected. The game ends when the selected elements generate the whole group S_n. The player who made the last move loses the game. The first move is made by A. Which player has a winning strategy?

A4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function that satises $f(t) > f(f(t))$ for all $t \in \mathbb{R}$. Prove that $f(f(f(t))) \leq 0$ for all $t \geq 0$.

A5. Let a be a rational number and let n be a positive integer. Prove that the polynomial $X^{2^n}(X + a)^{2^n} + 1$ is irreducible in the ring $\mathbb{Q}[X]$ of polynomials with rational coefficients.

B1. Consider a polynomial

$$f(x) = x^{2012} + a_{2011}x^{2011} + \cdots + a_1x + a_0.$$

Albert Einstein and Homer Simpson are playing the following game. In turn, they choose one of the coeffcients $a_0, \ldots, a_{2011}$ and assign a real value to it. Albert has the first move. Once a value is assigned to a coeffcient, it cannot be changed any more. The game ends after all the coefficients have been assigned values. Homers goal is to make $f(x)$ divisible by a fixed polynomial m(x) and Alberts goal is to prevent this.

(a) Which of the players has a winning strategy if $m(x) = x - 2012$?

(b) Which of the players has a winning strategy if $m(x) = x^2 + 1$?

B2. Define the sequence $a_0, a_1, \ldots$ inductively by $a_0 = 1, a_1 = \frac{1}{2}$ and

$$a_{n+1} = \frac{na_n^2}{1 + (n+1)a_n} \qquad \text{for } n \geq 1.$$

Show that the series $\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ converges and determine its value.

B3. Is the set of positive integers n such that $n! + 1$ divides $(2012n)!$ finite or infinite?

B4. Let $n \geq 2$ be an integer. Find all real numbers a such that there exist real numbers $x_1, \ldots, x_n$ satisfying

$$x_1(1 - x_2) = x_2(1 - x_3) = \cdots = x_{n-1}(1 - x_n) = x_n(1 - x_1) = a.$$

B5. Let $c \geq 1$ be a real number. Let G be an abelian group and let $A \subset G$ be a nite set satisfying $|A + A| \leq c|A|$, where $X + Y := \{x + y \mid x \in X, y \in Y\}$ and $|Z|$ denotes the cardinality of Z. Prove that

$$|A + A + \cdots + A| \leq c^k|A|$$

for every positive integer k.