17^{th} IMC Competition

2010

A1. Let $0 < a < b$. Prove that

$$\int_a^b (x^2+1)e^{-x^2}\,dx \geq e^{-a^2} - e^{-b^2}.$$

A2. Compute the sum of the series

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7\cdot 8} + \cdots .$$

A3. Dene the sequence $x_1, x_2, \ldots$ inductively by $x_1 = \sqrt{5}$ and $x_{n+1} = x_n^2 - 2$ for each $n \geq 1$. Compute

$$\lim_{n\to\infty} \frac{x_1 \cdot x_2 \cdot x_3 \cdots x_n}{x_{n+1}}.$$

B1. (a) A sequence $x_1, x_2, \ldots$ of real numbers satises

$$x_{n+1} = x_n \cos x_n$$

for all $n \geq 1$.

Does it follow that this sequence converges for all initial values x_1?

(b) A sequence $y_1, y_2, \ldots$ of real numbers satises

$$y_{n+1} = y_n \sin y_n$$

for all $n \geq 1$.

Does it follow that this sequence converges for all initial values $y1$?

B2. Let $a_0, a_1, \ldots, a_n$ be positive real numbers such that $a_{k+1} - a_k \geq 1$ for all $k = 0, 1, \ldots, n-1$. Prove that

$$1+\frac{1}{a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_n-a_0}\right) \leq \left(1+\frac{1}{a_0}\right)\left(1+\frac{1}{a_1}\right)\cdots\left(1+\frac{1}{a_n}\right).$$

B3. Denote by S_n the group of permutations of the sequence $(1, 2, \ldots, n)$. Suppose that G is a subgroup of S_n such that for every $\pi \in G \setminus \{e\}$ there exists a unique $k \in \{1, 2, \ldots, n\}$ for which $\pi(k) = k$. (Here e is the unit element in the group S_n.) Show that this k is the same for all $\pi \in G \setminus \{e\}$.