15^{th} IMC Competition

2008

A1. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) - f(y)$ is rational for all reals x and y such that $x - y$ is rational.

A2. Denote by V the real vector space of all real polynomials in one variable, and let $P : V \to \mathbb{R}$ be a linear map. Suppose that for all $f, g \in V$ with $P(fg) = 0$ we have $P(f) = 0$ or $P(g) = 0$. Prove that there exist real numbers x_0, c such that $P(f) = cf(x_0)$ for all $f \in V$.

A3. Let p be a polynomial with integer coefficients and let $a_1 < a_2 < ... < a_k$ be integers.

(a) Prove that there exists $a \in \mathbb{Z}$ such that $p(a_i)$ divides $p(a)$ for all $i = 1, 2, \ldots, k$.

(b) Does there exist an $a \in \mathbb{Z}$ such that the product $p(a_1)\cdot p(a_2)\cdots p(a_k)$ divides $p(a)$?

B1. Let n, k be positive integers and suppose that the polynomial $x^{2k} - x^k + 1$ divides $x^{2n} + x^n + 1$. Prove that $x^{2k} + x^k + 1$ divides $x^{2n} + x^n + 1$.

B2. Two different ellipses are given. One focus of the first ellipse coincides with one focus of the second ellipse. Prove that the ellipses have at most two points in common.

B3. Let n be a positive integer. Prove that $2n - 1$ divides

$$\sum_{0 \le k < n/2} \binom{n}{2k+1} 5^k.$$