14^{th} IMC Competition

2007

A1. Let f be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer k. Prove that all coefficients of f are divisible by 5.

A2. Let $n \geq 2$ be an integer. What is the minimal and maximal possible rank of an $n \times n$ matrix whose n^2 entries are precisely the numbers $1, 2, \dots, n^2$?

A3. Let C be a nonempty closed bounded subset of the real line and $f : C \to C$ be a nondecreasing continuous function. Show that there exists a point $p \in C$ such that $f(p) = p$.

(A set is closed if its complement is a union of open intervals. A function g is nondecreasing if $g(x) \leq g(y)$ for all $x \leq y$.)

B1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for any $c > 0$, the graph of f can be moved to the graph of cf using only a translation or a rotation. Does this imply that $f(x) = ax + b$ for some real numbers a and b?

B2. Let $x, y,$ and z be integers such that $S = x^4 + y^4 + z^4$ is divisible by 29. Show that S is divisible by 29^4.

B3. Call a polynomial $P(x_1, \dots, x_k)$ *good* if there exist 2×2 real matrices $A_1, \dots, A_k$ such that

$$P(x_1, \dots, x_k) = \det\left(\sum_{i=1}^{k} x_i A_i\right).$$

Find all values of k for which all homogeneous polynomials with k variables of degree 2 are good.

(A polynomial is homogeneous if each term has the same total degree.)

B4. Let $n > 1$ be an odd positive integer and $A = (a_{ij})_{i,j=1,\ldots,n}$ be the $n \times n$ matrix with

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i - j = \pm 2 \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

Find $\det A$.