9^{th} IMC Competition

2002

A1 A *standard* parabola has an equation of the form $y = x^2 + ax + b$. Three standard parabolas have vertices V_1, V_2, V_3 and intersect pairwise at the points A_1, A_2, A_3. $P \mapsto r(P)$ is reflection in the x-axis. Show that the standard parabolas with vertices $r(A_1), r(A_2), r(A_3)$ intersect pairwise at $r(V_1), r(V_2), r(V_3)$.

A2 Is there a function $f : \mathbb{R} \to \mathbb{R}$ with continuous derivative such that $f(x) > 0$ and $f'(x) = f(f(x))$ for all x?

A3 Put $a_n = 1/\binom{n}{k}$, $b_n = 1/2^{n-k}$ for $k = 1, 2, ..., n$ (where $\binom{n}{k}$ is the binomial coefficient). Show that

$$\sum \frac{a_i - b_i}{i} = 0.$$

A4 Let $f : [a, b] \to [a, b]$ be a continuous function. For $p \in [a, b]$ define $p_0 = p$, $p_{n+1} = f(p_n)$. The set $T_p = \{p_0, p_1, p_2, ...\}$ is closed. Show that it has only finitely many elements.

A5 Does there exist a monotonic function $f : [0, 1] \to [0, 1]$ such that $f(x) = k$ has uncountably many solutions for each $k \in [0, 1]$? Does there exist such a function which also has a continuous derivative?

A6 For a real $n \times n$ matrix M define $|M| = \sup_{x \neq 0} |Mx|/|x|$ (where $|x|$ is the standard Euclidean norm for $x \in R^n$). If the matrix A satisfies

$$|A^k - A^{-k}| \leq \frac{1}{2002k}$$

for all positive integers k, show that

$$|A^k| \leq 2002$$

for all k.

B1 The matrix $A = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} 2 \text{ if } i = j, \\ (-1)^{|}i - j| \text{ if } i \neq j. \end{cases}$$

Find $\det A$.

B2 200 students did an exam with 6 questions. Every question was correctly answered by at least 120 students. Show that there must be two students such that every question was correctly answered by at least one of them.

B3 Show that

$$\left(\sum_{k=0}^{\infty} k^n/k!\right) \left(\sum_{k=0}^{\infty} (-1)^k k^n/k!\right)$$

is an integer.

B4 $OABC$ is a tetrahedron. $\angle BOC = \alpha, \angle COA = \beta, \angle AOB = \gamma$. The angle between the faces OAB and OAC is σ, and the angle between faces OAB and OBC is τ. Show that $\gamma > \beta \cos \sigma + \alpha \cos \tau$.

B5 A is a complex $n \times n$ matrix for $n > 1$. A' is the complex conjugate of A (each element is the complex conjugate of the corresponding element of A). Show that $AA' = 1 \iff A = S(S')^{-1}$ for some S.

B6 $f : R^n \to R$ is convex. Δf exists at every point and for some $L > 0$ we have

$$|\Delta f(x_1) - \Delta f(x_2)| \leq L|x_1 - x_2|$$

for all x_1, x_2. Show that

$$|\Delta f(x_1) - \Delta f(x_2)|^2 \leq L(\Delta f(x_1) - \Delta f(x_2)).(x_1 - x_2)$$

(where '.' denotes the dot product).