8^{th} IMC Competition

2001

A1 The numbers from 1 to n^2 are entered in an $n \times n$ array, starting at the top left, moving along the top row, then left to right along the second row and so on. What are the possible values for the sum of n entries, one from each row and column?

A2 G is an abelian group. a and b are elements of G such that

$$a^m = b^n = (ab)^k$$

where m, n, k, are positive integers no two of which have a common factor. Show that $a = b = 1$.

Is this necessarily true in a non-abelian group?

A3 Find

$$\lim(1-t)\left(\frac{t}{1+t} + \frac{t^2}{1+t^2} + \frac{t^3}{1+t^3} + \ldots\right),$$

where the limit is taken as t approaches 1 from below.

A4 $p(x)$ is a polynomial of degree n with every coefficient 0 or ± 1, and $p(x)$ is divisible by $(x-1)^k$ for some integer $k > 0$. q is a prime such that

$$\frac{q}{\ln q} < \frac{k}{\ln(n+1)}.$$

Show that the complex qth roots of unity must be roots of $p(x)$.

A5 A is an $n \times n$ matrix, which is not a (complex) multiple of the identity matrix. Show that there are matrices B, C such that $A = BCB^{-1}$, where C has at most one non-zero diagonal entry.

A6 f, g, a, b are real-valued differentiable functions on the reals such that:

1. $f'(x)/g'(x) + a(x)f(x)/g(x) = b(x)$;
2. $f(x)$ and $g(x) \to \infty$ as $x \to \infty$
3. $a(x) \to A > 0$, and $b(x) \to B > 0$ as $x \to \infty$. Show that $f(x)/g(x) \to B/(A+1)$ as $x \to \infty$.

B1 a_i and b_j are non-negative reals such that

$$(a_0+a_1x+a_2x^2+...+a_{n-1}x^{n-1}+x^n)(b_0+b_1x+b_2x^2+...+b_{m-1}x^{m-1}+x^m)$$
$$= 1 + x + x^2 + ... + x^{m+n}.$$

Show that all a_i and b_j are 0 or 1.

B2 The sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are defined by

$$a_0 = \sqrt{2}, b_0 = 2, a_{n+1} = \sqrt{(2 - \sqrt{(4 - a_n^2)})}, b_{n+1} = 2b_n/(2 + \sqrt{(4 + b_n^2)}).$$

Show that both sequences are decreasing and converge to 0. Show that $2^n a_n$ increases to a limit, and that $2^n b_n$ decreases to the same limit. Show that $0 < b_n - a_n < c/8^n$ for some constant c.

B3 Find the largest number of points on a sphere radius 1 in R^n such that the distance between any two exceeds $\sqrt{2}$.

B4 A is a complex $n \times n$ matrix such that the determinant of the matrix formed by any m rows and m columns of A is zero (for $m = 1, 2, 3, \ldots, n$).

Show that $A^n = 0$, and that we can apply a permutation to the rows and the same permutation to the columns so that the resulting matrix has all the elements on or below the diagonal zero.

B5 Show that there is no real-valued function $f(x)$ on the reals such that

1. $f(x + y) \geq f(x) + yf(f(x))$ for all x, y and
2. $f(0) > 0$.

B6 Show that

$$|\sin x \sin 2x \sin 4x \cdots \sin 2^n x| \leq 2\sqrt{3}(\sin k \sin 2k \sin 4k \cdots \sin 2^n k)$$

for all x, where $k = \pi/3$.