7^{th} IMC Competition

2000

A1 Does every monotone increasing function $f : [0,1] \to [0,1]$ have a fixed point? What about every monotone decreasing function?

A2 $p(x) = x^5 + x,\ q(x) = x^5 + x^2$. Find all pairs (w, z) of complex numbers with $w \neq z$ such that $p(w) = p(z)$ and $q(w) = q(z)$.

A3 A, B are square complex matrices and $\operatorname{rank}(AB - BA) = 1$. Show that $(AB - BA)^2 = 0$.

A4 Show that if (x_i) is a decreasing sequence of positive reals then

$$\left(\sum_1^n x_i^2\right)^{1/2} \leq \sum_1^n \frac{x_i}{\sqrt{i}}.$$

A5 R is a ring of characteristic zero. e, f, g are elements of R such that $e + f + g = 0, e^2 = e, f^2 = f, g^2 = g$. Show that $e = f = g = 0$.

A6 $f : R \to (0, \infty)$ is an increasing differentiable function such that $f(x) \to \infty$ as $x \to \infty$, and f' is bounded. Let $F(x) = \int_0^x f(t)dt$. Define

$$a_0 = 1,\ a_{n+1} = a_n + 1/f(a_n) \text{ and } b_n = F^{-1}(n).$$

Prove that $\lim_{n\to\infty}(a_n - b_n) = 0$.

B1 Show that a square may be partitioned into n smaller squares for sufficiently large n. Show that for some constant $N(d)$, a d-dimensional cube can be partitioned into n smaller cubes if $n \geq N(d)$.

B2 f is continuous on $[0,1]$. There is no open subinterval of $[0,1]$ on which f is monotone. Show that the set of points on which f attains a local minimum is dense in $[0,1]$.

B3 $p(z)$ is a polynomial of degree $n > 0$ with complex coefficients. Show that $p(z)$ is 0 or 1 for at least $n + 1$ complex numbers z.

B4 The graph of a polynomial of degree 6 is tangent to a straight line at A, B and C where B lies between A and C. If B is the midpoint of AC show that the area bounded by AB and the graph equals the area bounded by BC and the graph. If $BC/AC = k$, show that the ratio K of these areas satisfies

$$2k^5/7 < K < 7k^5/2.$$

B5 R^+ is the set of positive real numbers. Find all functions $f : R^+ \to R^+$ such that $f(x)f(yf(x)) = f(x + y)$ for all x, y.

B6 For any $m \times m$ real matrix A, define $e^A = \sum_0^\infty A^n/n!$. Prove or disprove that for any real polynomial $p(x)$, $p(e^{AB})$ is nilpotent iff $p(e^{BA})$ is nilpotent.