6^{th} IMC Competition

1999

A1 Show that there is a real $m \times m$ matrix A such that $A^3 = A + I$, and show that it must satisfy $\det A > 0$.

A2 Is there a bijection f on the positive integers such that $\sum f(n)/n^2$ converges?

A3 $f : R \to R$ satisfies

$$|\sum_{k=1}^{n} 3k\,(f(x+ky) - f(x-ky))| \leq 1$$

for all x, y and all positive integers n. Show that it must be constant.

A4 Find all strictly monotonic functions $f : (0, \infty) \to (0, \infty)$ such that $f(x^2/f(x)) = x$ for all x.

A5 $2n$ points in an $n \times n$ array are colored red. Show that one can select an even number of red points a_i such that a_{2i}, a_{2i+1} are in the same column for $i \leq n$ (taking a_{2n+1} to mean a_1) and a_{2i-1}, a_{2i} are in the same row.

A6 Let S be the set of functions $f : [-1, 1] \to R$ which have continuous derivatives such that $f(1) > f(-1)$ and $|f'(x)| \leq 1$ for all x. Show that for $1 < p < \infty$ we can find a constant c_p, such that given $f \in S$ we can find x_0 such that $|f'(x_0)| > 0$ and

$$|f(x) - f(x_0)| \leq c_p f'(x0)^{1/p} |x - x_0|$$

for all x. Does c_1 exist?

B1 R is a ring. For every $a \in R$, $a2 = 0$. Show that $abc + abc = 0$ for any $a, b, c \in R$.

B2 A fair die is thrown 10 times. What is the probability that the total is divisble by 5?

B3 $x_1, x_2, \ldots, x_n$ are reals ≥ -1 such that $\sum x_i^3 = 0$. Show that $\sum x_i \leq n/3$.

B4 Show that no function $f : (0, \infty) \to (0, \infty)$ satisfies $f(x)f(x) \leq f(x + y)(f(x) + y)$ for all x, y.

B5 S is the set of all words made from the letters x, y, z. The equivalence relation $\sim$ on S satisfies

1. $uu \sim u$,
2. if $u \sim v$, then $uw \sim uw$ and $wu \sim wv$.

Show that every word is equivalent to a word of length ≤ 8.

B6 A is a subset of $Z/(n)$ (the integers mod n) with at most $(\ln n)/100$ elements. Define $f(r) = \sum_{s \in A} e^{2isr/n}$. Show that for some $r \neq 0$ we have $|f(r)| \geq |A|/2$.