Problem Solving (MA2201)

Week 9

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1. A domino covers 2 squares on a chess-board. If two opposite corner squares on the board are removed, show that it is not possible to cover the remaining 62 squares with 31 dominoes.

2. Find all polynomials $f(x)$ such that
$$f(x^2) = f(x)^2.$$

3. Show that if A, B, C are the angles of a triangle then
$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

4. Prove that
$$\sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=0}^{2n-1} \frac{(-1)^k}{k+1}.$$

5. Does there exist a non-zero polynomial $f(x, y)$ such that
$$f([x], [2x]) = 0$$
for all real x. (Recall that $[x]$ is the largest integer $\leq x$.)

6. Evaluate
$$\int_0^1 \frac{\log(x+1)}{x^2+1} dx.$$

7. Given a point O and a line ℓ in the plane, what is the locus of a point P which moves so that the sum of its distances from O and ℓ is constant?

8. Show that if $a_n > 0$ and $\lim_{n\to\infty} a_n = 0$ then the equation

$$a_i + a_j + a_k = 1$$

holds only for a finite number of triples i, j, k.

9. In how many different ways can $2n$ points on the circumference of a circle. be joined in pairs by n cords which do not intersect within the circle?

10. A hole of diameter 1 is drilled through the centre of a sphere of radius 1. What is the volume of the remaining material?

11. Solve the simultaneous equations

$$\begin{aligned} x + y + z &= 2 \\ x^2 + y^2 + z^2 &= 5 \\ x^3 + y^3 + z^3 &= 8. \end{aligned}$$

12. Show that for any positive integers $m \leq n$ the sum

$$\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n},$$

when expressed in its lowest terms, has odd numerator.

13. The function $f(x)$ satisfies $f(0) = 1,\ f'(0) = 0$ and

$$(1 + f(x))f"(x) = 1 + x$$

for all real x. Determine the maximum value of $f'(1)$, and the maximum and minimum values of $f'(-1)$.

14. A group G is a union of 3 proper subgroups if and only if there is a surjective homomorphism $G \to K$ where K is the Klein 4-group.

15. Find all solutions in integers of the equation

$$x^2 = y^3 + 1.$$

Challenge Problem

Let $a_1 = 1/2,\ a_{n+1} = a_n - a_n^2$. Find a real number c for which the sequence $b_n = n^c a_n$ has a finite limit, and determine this limit.