Problem Solving (MA2201)

Week 9

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1. A domino covers 2 squares on a chess-board. If two opposite corner squares on the board are removed, show that it is not possible to cover the remaining 62 squares with 31 dominoes.

   Answer: *The squares will be the same colour, say white, so there will be 32 black squares left but only 30 white. Each domino covers one black and one white square; hence there is no way of covering them all.*

2. Find all polynomials $f(x)$ such that
$$f(x^2) = f(x)^2.$$

   Answer: *Suppose* $f(x)$ *has just one term, say*
$$f(x) = cx^n$$
   Then
$$c^2 = c \implies c = 1$$
   (if $f(x) \neq 0$*).*

   Now suppose $f(x)$ *has more than one term. Let the first two terms be*
$$f(x) = a_n x^n + a_m x^m + \cdots .$$
   Then
$$f(x^2) = a_n x^{2n} + a_m x^{2m} + \cdots ,$$

while the first two terms of the square are

$$f(x)^2 = a_n^2 x^{2n} + 2a_n a_m x^{n+m} + \cdots .$$

Hence

$$2m = m + n,$$

which is impossible.

Hence the only non-zero solutions are

$$f(x) = x^n.$$

3. Show that if A, B, C are the angles of a triangle then

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Answer: *Since* $C = \pi - (A + B)$,

$$\begin{aligned}\tan C &= -\tan(A + B)\\ &= -\frac{\tan A + \tan B}{1 - \tan A \tan B}.\end{aligned}$$

Hence

$$\tan C - \tan A \tan B \tan C = -(\tan A + \tan B),$$

ie

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

4. Prove that

$$\sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=0}^{2n-1} \frac{(-1)^k}{k+1}.$$

Answer: *If* $n = 1$ *the equation reads*

$$\frac{1}{2} = 1 - \frac{1}{2},$$

while if $n = 2$ *it reads*

$$\frac{1}{3} + \frac{1}{4} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}.$$

Let us try to proved the identity by induction. We have seen that it is true for $n = 1$. Suppose it is true for $n - 1$.

On passing to n we add

$$-\frac{1}{n} + \frac{1}{2n-1} + \frac{1}{2n} = \frac{1}{2n} + \frac{1}{2n-1}$$

to the left-hand side, while

$$\frac{1}{2n-1} + \frac{1}{2n}$$

is added to the right-hand side.

Since these are equal, it follows by induction that the identity holds for all n.

5. Does there exist a non-zero polynomial $f(x, y)$ such that

$$f([x], [2x]) = 0$$

for all real x. (Recall that $[x]$ is the largest integer $\leq x$.)

Answer: *If*

$$g(x, y) = y - 2x$$

then

$$g([x], [2x]) = \begin{cases} 0 \text{ if } n \leq x < n + 1/2, \\ 1 \text{ if } n + 1/2 \leq x < n + 1. \end{cases}$$

Thus if

$$h(x, y) = 2y - x - 1$$

then

$$g([x], [2x]) = \begin{cases} -1 \text{ if } n \leq x < n + 1/2, \\ 0 \text{ if } n + 1/2 \leq x < n + 1. \end{cases}$$

Hence

$$f(x, y) = g(x, y)h(x, y)$$

has the required property.

6. Evaluate
$$\int_0^1 \frac{\log(x+1)}{x^2+1} dx.$$

Answer: *Integrating by parts, the integral*

$$I = S - J,$$

where

$$\begin{aligned} S &= [\log(x+1)\arctan(x)]_0^1 \\ &= \log 2 \cdot \frac{\pi}{4}, \\ J &= \int_0^1 \frac{\arctan x}{x+1} dx \end{aligned}$$

7. Given a point O and a line ℓ in the plane, what is the locus of a point P which moves so that the sum of its distances from O and ℓ is constant?

Answer: *Take the line through O parallel to ℓ, and the line through O perpendicular to this, as coordinate axes.*

Then
$$OP = \sqrt{x^2+y^2},$$
On the other hand, the distance of O from ℓ is

$$d + y,$$

where d is the distance from O to ℓ.

Thus the locus is given by the equation

$$\sqrt{x^2+y^2} + d + y = c,$$

ie

$$x^2 + y^2 = (c - y - d)^2.$$

This reduces to

$$\begin{aligned} x^2 &= 2(d-c)y + (c+d)^2 \\ &= 4ay_1, \end{aligned}$$

where $a = (d - c)/2$ *and*

$$y_1 = y + (c + d)^2/4a.$$

Thus the locus is a parabola, symmetric about the y*-axis.*

8. Show that if $a_n > 0$ and $\lim_{n\to\infty} a_n = 0$ then the equation

$$a_i + a_j + a_k = 1$$

holds only for a finite number of triples i, j, k.

Answer:

9. In how many different ways can $2n$ points on the circumference of a circle. be joined in pairs by n cords which do not intersect within the circle?

Answer: *This is a nice exercise in generating polynomials. Let the number of ways of joining* $2n$ *points in this way be* a_n.

Suppose the points are $P_1, P_2, \ldots, P_{2n}$ *in that order around the circle. Let us choose one point, say* P_1.

It is easy to see that there are an even number of points between 2 points joined by a chord. So P_1 *can be joined to any of the points* $P_2, P_4, \ldots, P_{2n}$.

Suppose P_1 *is joined to* P_{2r}. *There are* $2(r-1)$ *points between* P_1 *and* P_{2r}*, and it is clear that these points are joined in accordance with the same rules. Thus these points can be joined in* a_{r-1} *ways.*

Similarly the $2(n-r-1)$ *points between* P_{2r} *and* P_1 *(going the same way round the circle) can be joined in* a_{n-r-1} *ways.*

Thus there are $a_{r-1}a_{n-r-1}$ *ways of joining the remaining points if* P_1 *is joined to* P_{2r}. *(This also holds if* P_1 *is joined to an adjacent point,* P_2 *or* P_{2n}*, provided we set* $a_0 = 1$.*)*

Adding the contributions from the different end-points to the chord from P_1,

$$a_n = a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_{n-1} a_0,$$

for $n \geq 1$.

Now let us introduce the generating function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots .$$

The relation above translates into

$$f(x) - 1 = x f(x)^2,$$

ie

$$x f(x)^2 - f(x) + 1 = 0.$$

Solving this quadratic equation for $f(x)$*,*

$$f(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

Since $f(x)$ *is a power-series in* x*, we must take the negative sign:*

$$f(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

By the binomial theorem

$$\begin{aligned}(1 - 4x)^{1/2} &= 1 - (1/2)4x + \frac{(1/2)(-1/2)}{2!}(4x)^2 - \frac{(1/2)(-1/2)(-3/2)}{3!}(4x)^3 - \\ &= 1 - 2x - \frac{1}{2!}(2x)^2 - \frac{1 \cdot 3}{3!}(2x)^3 - \cdots .\end{aligned}$$

Thus

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{(n + 1)!} 2^n.$$

10. A hole of diameter 1 is drilled through the centre of a sphere of radius 1. What is the volume of the remaining material?

    Answer:

11. Solve the simultaneous equations

$$\begin{aligned} x + y + z &= 2 \\ x^2 + y^2 + z^2 &= 5 \\ x^3 + y^3 + z^3 &= 8. \end{aligned}$$

Answer: *Note that x, y, z are the roots of the equation*

$$t^3 - at^2 + bt - c,$$

where

$$a = \sum x = 2,\ b = \sum xy,\ c = xyz.$$

Now

$$(\sum x)^2 = \sum x^2 + 2\sum xy.$$

Thus

$$\sum xy = 1/2(4-5) = -1/2.$$

Similarly

$$(\sum x)^3 = \sum x^3 + 3\sum x^2y + 6xyz.$$

But

$$\sum x \sum xy = \sum x^2y + 3xyz.$$

Hence

$$\sum x^2y = \sum x \sum xy - 3xyz,$$

and so

$$(\sum x)^3 = \sum x^3 + 3\sum x \sum xy - 3xyz.$$

Thus

$$xyz = -1.$$

The cubic is therefore

$$t^3 - 2t^2 - 1/2t + 1 = 0,$$

ie

$$2t^3 - 4t^2 - t + 2 = 0.$$

By observation, this has solution $t = 2$. Dividing by $t - 2$,

$$2t^3 - 4t^2 - t + 2 = (t-2)(2t^2 - 1).$$

Thus the roots are $2, \pm\frac{1}{\sqrt{2}}$; and the solution to the equations is

$$\{x, y, z\} = \{2, \pm\frac{1}{\sqrt{2}}\}.$$

12. Show that for any positive integers $m \le n$ the sum

$$\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n},$$

when expressed in its lowest terms, has odd numerator.

Answer:

13. The function $f(x)$ satisfies $f(0) = 1,\ f'(0) = 0$ and

$$(1 + f(x))f''(x) = 1 + x$$

for all real x. Determine the maximum value of $f'(1)$, and the maximum and minimum values of $f'(-1)$.

Answer:

14. A group G is a union of 3 proper subgroups if and only if there is a surjective homomorphism $G \to K$ where K is the Klein 4-group.

Answer:

15. Find all solutions in integers of the equation

$$x^2 = y^3 + 1.$$

Answer:

Challenge Problem

Let $a_1 = 1/2,\ a_{n+1} = a_n - a_n^2$. Find a real number c for which the sequence $b_n = n^c a_n$ has a finite limit, and determine this limit.

Answer: *By induction*

$$0 < a_n \le \frac{1}{2}.$$

Since a_n is decreasing, it follows that a_n converges to a limit ℓ.

From the defining relation,

$$\ell = \ell - \ell^2 \implies \ell = 0,$$

ie

$$a_n \to 0.$$

It is helpful to turn to an analogical problem from differential calculus, namely: Solve the differental equation

$$\frac{dy}{dx} = -y^2.$$

Here we are replacing n *with* x*, the sequence* a_n *with the function* $y(x)$*, and* $a_{n+1} - a_n$ *with* dy/dx*.*

This equation can be written

$$-\frac{dy}{y^2} = dx,$$

with the solution

$$\frac{1}{y} = x + c,$$

ie

$$y = \frac{1}{x+c}.$$

So let us try

$$b_n = \frac{1}{n+1},$$

so that

$$b_1 = \frac{1}{2} = a_1.$$

Also

$$a_2 = 1/4, b_2 = 1/3, a_3 = 3/16, b_3 = 1/4, a_4 = 39/256, b_4 = 1/5.$$

By the Mean Value Theorem,

$$b_{n+1} - b_n = -\frac{1}{(n+\theta)^2},$$

where $1 < \theta < 2$*. Thus*

$$\frac{1}{(n+1)^2} < b_n - n_{n+1} < \frac{1}{n^2},$$

ie

$$b_{n+1}^2 < b_n - b_{n+1} < b_n^2.$$

As we shall see, it follows from this by induction that

$$a_n \leq b_n.$$

This holds for $n = 1$. *Suppose it holds for* $1 \leq r \leq n$. *Then*

$$a_r - a_{r+1} = a_r^2 \leq b_r^2 < b_{r-1} - b_r.$$

Adding these inequalities for $r = 1, 2, \ldots, n$,

$$a_1 - an + 1 \leq b_1 - b_{n+1},$$

and so

$$a_{n+1} \leq b_{n+1}.$$

Similarly