

Problem Solving (MA2201)

Week 8

Timothy Murphy

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1. Show that there must be 2 people at a party who know the same number of people at the party.
2. If a set S of circles in the plane has the property that each point P in the plane lies in only a finite number of the circles, does it follow that S is enumerable?
3. Show that there are an infinite number of distinct positive integers a, b such that both ab and $(a + 1)(b + 1)$ are perfect squares.
4. Determine the 100th derivative of the function

$$\frac{x^2 + 1}{x^3 - x}.$$

5. Find all positive integers a, b satisfying

$$1 + 2^a = 3^b.$$

6. What fraction of the volume of a hypercube in 5 dimensions is taken by an inscribed sphere?
7. A man makes 45 phone-calls in 30 days. He makes at least one call each day. Show that there is a succession of days on which he makes 14 calls.

8. Solve the differential equation

$$\frac{dy}{dx} = x + \frac{x^3}{y}.$$

9. For what value (or values) of c is the line $y = 10x$ tangent to the curve $y = e^{cx}$ at some point in the xy -plane?

10. Suppose $P(x), Q(x)$ are two non-constant real polynomials such that

$$P(x)^n - 1 \mid Q(x)^n - 1$$

for all $n \in \mathbb{N}$. Does it follow that $Q(x) = P(x)^k$ for some k ?

11. Alice and Bob take turns to fill in entries in a 100×100 matrix. The matrix has no entries initially, and Alice goes first. If when all of the entries are filled in, the determinant of the matrix is 0, then Bob wins; otherwise Alice wins. What is the result if each adopts the best strategy?

12. Show that

$$\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}.$$

13. Show that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{3}{4}.$$

14. There are 25 men and 25 women sitting around a table. Show that some person is sitting between 2 women.

15. Determine

$$\int \sec \theta d\theta.$$

Challenge Problem

Suppose $f(P)$ is a real-valued function on the plane such that

$$f(A) + f(B) + f(C) + f(D) = 0$$

for every square $ABCD$ in the plane. Does it follow that $f(P) = 0$ for all points P ?