Problem Solving (MA2201)

Week 8

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1. Show that there must be 2 people at a party who know the same number of people at the party.

   Answer:

2. If a set S of circles in the plane has the property that each point P in the plane lies in only a finite number of the circles, does it follow that S is enumerable?

   Answer:

3. Show that there are an infinite number of distinct positive integers a, b such that both ab and $(a+1)(b+1)$ are perfect squares.

   Answer: *Let us take*
   $$a = c,\ b = ct^2.$$
   Then
   $$ab = (ct)^2$$
   is a perfect square, and we have to find t *such that*
   $$(c+1)(ct^2+1) = u^2,$$
   ie
   $$u^2 - c(c+1)t^2 = c+1.$$

Let us try $c = 1$. We have to solve the Pell-like equation

$$u^2 - 2t^2 = 2.$$

This has the trivial solution $(u,t) = (2,1)$, arising from $a = b = 1$. That solution is invalid; but we know that the equation has an infinity of solutions if it has one.

For

$$(u^2 - 2t^2)(U^2 - 2T^2) = (uU + 2tT)^2 - 2(uT - tU)^2.$$

Thus any solution of Pell's equation

$$U^2 - 2T^2 = 1$$

will give a solution of our equation.

Again, this Pell's equation has the simple solution $(U,T) = (3,2)$; and the identity above enables us to get an infinity of solutions.

It is easier to express this in terms of square roots, using

$$(u - \sqrt{2}t)(u + \sqrt{2}t) = u^2 - 2t^2.$$

It follows that we get a solution

$$U + \sqrt{2}T = (3 + 2\sqrt{2}t)^e$$

to Pell's equation for each power e, and this gives us a solution

$$u + \sqrt{t} = (2 + \sqrt{2})(U + T\sqrt{2})$$

For example,

$$(3 + 2\sqrt{2}) = 17 + 12\sqrt{2}$$

gives $(U,T) = (17,12)$; and since

$$(2 + \sqrt{2})(17 + 12\sqrt{2}) = 58 + 39\sqrt{2},$$

this gives the solution

$$ab = 1 \cdot 1521 = 39^2, \ (a+1)(b+1) = 2 \cdot 1522 = 58^2.$$

4. Determine the 100th derivative of the function

$$\frac{x^2+1}{x^3-x}.$$

Answer: *Let*

$$f(x) = \frac{x^2+1}{x^3-x}.$$

Since

$$x^3 - x = x(x-1)(x+1),$$

we can express $f(x)$ as partial fractions

$$f(x) = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{x+1}.$$

Multiplying by x and setting $x = 0$,

$$a = -1.$$

Multiplying by $x-1$ and setting $x = 1$,

$$b = \frac{3}{2}.$$

Multiplying by $x+1$ and setting $x = -1$,

$$c = \frac{3}{2}.$$

Thus

$$f(x) = -\frac{1}{x} + \frac{3}{2}\left(\frac{1}{x-1} + \frac{1}{x+1}\right).$$

The nth derivative of

$$\frac{1}{x+c}$$

is

$$\frac{(-1)^n n!}{(x+c)^{n+1}}.$$

Hence the 100th derivative of $f(x)$ is

$$\frac{100!}{2}\left(-\frac{2}{x^{100}} + \frac{3}{(x-1)^{100}} + \frac{3}{(x+1)^{100}}\right).$$

5. Find all positive integers a, b satisfying

$$1 + 2^a = 3^b.$$

Answer: *There are 2 obvious solutions:*

$$1 + 2 = 3 \textit{ and } 1 + 2^3 = 3^2;$$

Omitting these cases, we may assume that $a \geq 4,\ b \geq 3$
If $a \geq 2$ *then*

$$3^b \equiv 1 \bmod 4.$$

Hence b *is even.*

6. What fraction of the volume of a hypercube in 5 dimensions is taken by an inscribed sphere?

Answer: *We must determine the volume* $V(n)$ *of a unit sphere in* n *dimensions.*

We have

$$V(n) = \int_{x_1^2+\cdots+x_n^2\leq 1} dx_1 \cdots dx_n.$$

Integrating first over $x_1, \ldots, x_{n-1}$ *for each* x_n*, and then over* x_n*,*

$$\begin{aligned} V(n) &= \int_{-1}^{1} \left(\int_{x_1^2+\cdots+x_{n-1}^2\leq 1-x_n^2} dx_1 \cdots dx_{n-1} \right) dx_n \\ &= \int_{-1}^{1} V(n-1)(1-x_n^2)^{(n-1)/2} dx_n, \end{aligned}$$

since a sphere of radius r *in* n *dimensions has volume* $V(n)r^n$*.*

Setting $x_n = \sin\theta$*,*

$$\begin{aligned} V(n) &= 2V(n-1) \int_0^{\pi/2} cos^{n-1}\theta \cos\theta d\theta \\ &= 2V(n-1) \int_0^{\pi/2} cos^{n}\theta d\theta \\ &= 2V(n-1)I(n), \end{aligned}$$

where

$$I(n) = \int_0^{\pi/2} cos^n\theta d\theta.$$

Now

$$I(n) = \int_0^{\pi/2} cos^{n-2}\theta(1 - \sin^2\theta)d\theta.$$

On integrating by parts,

$$\begin{aligned}\int_0^{\pi/2} cos^{n-2}\theta \sin^2\theta d\theta &= \int_0^{\pi/2} (cos^{n-2}\theta \sin\theta) \sin\theta d\theta \\ &= \left[\frac{\cos^{n-1}\theta}{n-1}\sin\theta\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\cos^{n-1}\theta}{n-1}\cos\theta d\theta \\ &= -\frac{I(n)}{n-1}.\end{aligned}$$

Thus

$$I(n) = I(n-2) + I(n)/(n-1),$$

ie

$$I(n) = \frac{n-1}{n-2}\, I(n-2)$$

Since

$$V(1) = 2,\ V(2) = \pi,$$

it follows that

$$V(n) = \begin{cases} \frac{(n-1)(n-3)\cdots 1}{(n-2)(n-4)\cdots 2}\, \pi \text{ if } n \text{ is even,} \end{cases}$$

7. A man makes 45 phone-calls in 30 days. He makes at least one call each day. Show that there is a succession of days on which he makes 14 calls.

Answer:

8. Solve the differential equation

$$\frac{dy}{dx} = x + \frac{x^3}{y}.$$

Answer: *Suppose*

$$y = cx^2.$$

Then

$$2cx = x + \frac{x}{c}.$$

Thus

$$2c = 1 + \frac{1}{c},$$

ie

$$2c^2 - c - 1 = 0,$$

ie

$$(2c + 1)(c - 1) = 0,$$

ie

$$c = 1 \ \ or \ \ -\frac{1}{2}.$$

We have

$$y\frac{dy}{dx} = xy + x^3.$$

Let

$$y = x^3 z.$$

Then

$$\frac{dy}{dx} = 3x^2 z + x^3 \frac{dz}{dx}.$$

Thus

$$\frac{dz}{dx} = -\frac{3z}{x} + x + \frac{1}{z}.$$

9. For what value (or values) of c is the line $y = 10x$ tangent to the curve $y = e^{cx}$ at some point in the xy-plane?

Answer: *We have*

$$\frac{dy}{dx} = ce^{cx} = 10$$

when

$$cx = \log(10/c),$$

ie

$$x = frac\log 10 - \log cc.$$

At this point

$$y = \frac{10}{c},$$

and the tangent to the curve is

$$y - \frac{1}{c} = 10\left(x - frac\log 10 - \log cc\right).$$

This passes through the origin if

$$\frac{1}{c} = 10frac\log 10 - \log cc,$$

ie

$$\frac{1}{10} = \log 10 - \log c,$$

Thus

$$\log c = \log 10 - \frac{1}{10},$$

ie

$$c = 10e^{1/10}.$$

10. Suppose $P(x), Q(x)]$ are two non-constant real polynomials such that
$$P(x)^n - 1 \mid Q(x)^n - 1$$
for all $n \in \mathbb{N}$. Does it follow that $Q(x) = P(x)^k$ for some k?

Answer:

11. Alice and Bob take turns to fill in entries in a 100×100 matrix. The matrix has no entries initially, and Alice goes first. If when all of the entries are filled in, the determinant of the matrix is 0, then Bob wins; otherwise Alice wins. What is the result if each adopts the best strategy?

Answer:

12. Show that
$$\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}.$$

Answer:

13. Show that
$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{3}{4}.$$

Answer:

14. There are 25 men and 25 women sitting around a table. Show that some person is sitting between 2 women.

Answer:

15. Determine
$$\int \sec \theta d\theta.$$

Answer:

Challenge Problem

Suppose $f(P)$ is a real-valued function on the plane such that

$$f(A) + f(B) + f(C) + f(D) = 0$$

for every square $ABCD$ in the plane. Does it follow that $f(P) = 0$ for all points P?

Answer: *Draw a square $ABCD$. Split it into 4 squares of half the size. Let the mid-points of AB, BC, CD, DA be X, Y, Z, T, and let the centre of the square be O.*

From the identity for square $AXOT$,

$$f(A) + f(X) + f(O) + f(T) = 0.$$

From the identity for square $ZDTO$,

$$f(Z) + f(X) + f(T) + f(O) = 0.$$

Subtracting the second from the first,

$$f(A) + f(X) - f(Z) + f(D) = 0,$$

ie

$$f(A) - f(D) = -(f(X) - f(O)).$$

By the same argument, applied to the lower 2 squares,

$$f(X) - f(O) = -(f(B) - f(C)).$$

Thus

$$f(A) - f(D) = f(B) - f(C),$$

ie

$$f(A) - f(B) + f(C) - f(D) = 0.$$

Adding the identity

$$f(A) + f(B) + f(C) + f(D) = 0$$

for the whole square,

$$2(f(A) + f(C)) = 0,$$

ie

$$f(C) = -f(A).$$

This hold for any 2 (different) points, since we can construct a square with any two points as diagonally opposite vertices. But then, taking any other point, say B, we have

$$f(A) = -f(B) = f(C).$$

Hence

$$f(A) = 0$$

for any point A.