

Problem Solving (MA2201)

Week 7

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1. Show that there are an infinite number of distinct positive integers a, b such that both ab and $(a+1)(b+1)$ are perfect squares.

Answer: *Let us take*

$$a = c, b = ct^2.$$

Then

$$ab = (ct)^2$$

is a perfect square, and we have to find t such that

$$(c+1)(ct^2+1) = u^2,$$

ie

$$u^2 - c(c+1)t^2 = c+1.$$

Let us try $c = 1$. We have to solve the Pell-like equation

$$u^2 - 2t^2 = 2.$$

This has the trivial solution $(u, t) = (2, 1)$, arising from $a = b = 1$. That solution is invalid; but we know that the equation has an infinity of solutions if it has one.

For

$$(u^2 - 2t^2)(U^2 - 2T^2) = (uU + 2tT)^2 - 2(uT - tU)^2.$$

Thus any solution of Pell's equation

$$U^2 - 2T^2 = 1$$

will give a solution of our equation.

Again, this Pell's equation has the simple solution $(U, T) = (3, 2)$; and the identity above enables us to get an infinity of solutions.

It is easier to express this in terms of square roots, using

$$(u - \sqrt{2}t)(u + \sqrt{2}t) = u^2 - 2t^2.$$

It follows that we get a solution

$$U + \sqrt{2}T = (3 + 2\sqrt{2})^e$$

to Pell's equation for each power e , and this gives us a solution

$$u + \sqrt{t} = (2 + \sqrt{2})(U + T\sqrt{2})$$

For example,

$$(3 + 2\sqrt{2}) = 17 + 12\sqrt{2}$$

gives $(U, T) = (17, 12)$; and since

$$(2 + \sqrt{2})(17 + 12\sqrt{2}) = 58 + 39\sqrt{2},$$

this gives the solution

$$ab = 1 \cdot 1521 = 39^2, \quad (a + 1)(b + 1) = 2 \cdot 1522 = 58^2.$$

2. Determine the 100th derivative of the function

$$\frac{x^2 + 1}{x^3 - x}.$$

Answer: *Let*

$$f(x) = \frac{x^2 + 1}{x^3 - x}.$$

Since

$$x^3 - x = x(x - 1)(x + 1),$$

we can express $f(x)$ as partial fractions

$$f(x) = \frac{a}{x} + \frac{b}{x - 1} + \frac{c}{x + 1}.$$

Multiplying by x and setting $x = 0$,

$$a = -1.$$

Multiplying by $x - 1$ and setting $x = 1$,

$$b = \frac{3}{2}.$$

Multiplying by $x + 1$ and setting $x = -1$,

$$c = \frac{3}{2}.$$

Thus

$$f(x) = -\frac{1}{x} + \frac{3}{2} \left(\frac{1}{x - 1} + \frac{1}{x + 1} \right).$$

The n th derivative of

$$\frac{1}{x + c}$$

is

$$\frac{(-1)^n n!}{(x + c)^{n+1}}.$$

Hence the 100th derivative of $f(x)$ is

$$\frac{100!}{2} \left(-\frac{2}{x^{100}} + \frac{3}{(x - 1)^{100}} + \frac{3}{(x + 1)^{100}} \right).$$

3. What fraction of the volume of a hypercube in 5 dimensions is taken by an inscribed sphere?

Answer: We must determine the volume $V(n)$ of a unit sphere in n dimensions.

We have

$$V(n) = \int_{x_1^2 + \dots + x_n^2 \leq 1} dx_1 \cdots dx_n.$$

Integrating first over x_1, \dots, x_{n-1} for each x_n , and then over x_n ,

$$\begin{aligned} V(n) &= \int_{-1}^1 \left(\int_{x_1^2 + \dots + x_{n-1}^2 \leq 1 - x_n^2} dx_1 \cdots dx_{n-1} \right) dx_n \\ &= \int_{-1}^1 V(n-1)(1 - x_n^2)^{(n-1)/2} dx_n, \end{aligned}$$

since a sphere of radius r in n dimensions has volume $V(n)r^n$.

Setting $x_n = \sin \theta$,

$$\begin{aligned} V(n) &= 2V(n-1) \int_0^{\pi/2} \cos^{n-1} \theta \cos \theta d\theta \\ &= 2V(n-1) \int_0^{\pi/2} \cos^n \theta d\theta \\ &= 2V(n-1)I(n), \end{aligned}$$

where

$$I(n) = \int_0^{\pi/2} \cos^n \theta d\theta.$$

Now

$$I(n) = \int_0^{\pi/2} \cos^{n-2} \theta (1 - \sin^2 \theta) d\theta.$$

On integrating by parts,

$$\begin{aligned}\int_0^{\pi/2} \cos^{n-2}\theta \sin^2 \theta d\theta &= \int_0^{\pi/2} (\cos^{n-2}\theta \sin \theta) \sin \theta d\theta \\ &= \left[\frac{\cos^{n-1} \theta}{n-1} \sin \theta \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\cos^{n-1} \theta}{n-1} \cos \theta d\theta \\ &= -\frac{I(n)}{n-1}.\end{aligned}$$

Thus

$$I(n) = I(n-2) + I(n)/(n-1),$$

ie

$$I(n) = \frac{n-1}{n-2} I(n-2)$$

Since

$$V(1) = 2, V(2) = \pi,$$

it follows that

$$V(n) = \begin{cases} \frac{(n-1)(n-3)\cdots 1}{(n-2)(n-4)\cdots 2} \pi & \text{if } n \text{ is even,} \end{cases}$$

4. Solve the differential equation

$$\frac{dy}{dx} = x + \frac{x^3}{y}.$$

Answer: Suppose

$$y = cx^2.$$

Then

$$2cx = x + \frac{x}{c}.$$

Thus

$$2c = 1 + \frac{1}{c},$$

ie

$$2c^2 - c - 1 = 0,$$

ie

$$(2c + 1)(c - 1) = 0,$$

ie

$$c = 1 \text{ or } -\frac{1}{2}.$$

We have

$$y \frac{dy}{dx} = xy + x^3.$$

Let

$$y = x^3 z.$$

Then

$$\frac{dy}{dx} = 3x^2 z + x^3 \frac{dz}{dx}.$$

Thus

$$\frac{dz}{dx} = -\frac{3z}{x} + x + \frac{1}{z}.$$

5. Determine

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2} dx$$

Answer:

6. For what value (or values) of c is the line $y = 10x$ tangent to the curve $y = e^{cx}$ at some point in the xy -plane?

Answer: We have

$$\frac{dy}{dx} = ce^{cx} = 10$$

when

$$cx = \log(10/c),$$

ie

$$x = \frac{\log 10}{c} - \log c.$$

At this point

$$y = \frac{10}{c},$$

and the tangent to the curve is

$$y - \frac{1}{c} = 10(x - \frac{\log 10}{c} - \log c).$$

This passes through the origin if

$$\frac{1}{c} = 10 \frac{\log 10}{c} - \log c,$$

ie

$$\frac{1}{10} = \log 10 - \log c,$$

Thus

$$\log c = \log 10 - \frac{1}{10},$$

ie

$$c = 10e^{1/10}.$$

7. If a set S of circles in the plane has the property that each point P in the plane lies in only a finite number of the circles, does it follow that S is enumerable?

Answer:

8. Show that there must be 2 people at a party who know the same number of people at the party.

Answer:

9. A man makes 45 phone-calls in 30 days. He makes at least one call each day. Show that there is a succession of days on which he makes 14 calls.

Answer:

10. Suppose $P(x), Q(x)$ are two non-constant real polynomials such that

$$P(x)^n - 1 \mid Q(x)^n - 1$$

for all $n \in \mathbb{N}$. Does it follow that $Q(x)$ is a power of $P(x)$ for some k ?

Answer:

11. Find all positive integers a, b satisfying

$$1 + 2^a = 3^b.$$

Answer:

12. Alice and Bob take turns to fill in entries in a 100×100 matrix. The matrix has no entries initially, and Alice goes first. If when all of the entries are filled in, the determinant of the matrix is 0, then Bob wins; otherwise Alice wins. What is the result if each adopts the best strategy?

Answer:

13. Show that

$$\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}.$$

Answer:

14. Suppose that $f(x)$ is increasing on $[0, 1]$, that $f(0) = 0$, and that $f'(x)$ exists and is increasing on $(0, 1)$. Show that $f(x)/x$ is increasing on $[0, 1]$

Answer:

15. Show that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{3}{4}.$$

Answer:

16. There are 25 men and 25 women sitting around a table. Show that some person is sitting between 2 women.

Answer:

17. Determine

$$\int \sec \theta d\theta.$$

Answer:

Answer:

18. Does there exist a polynomial

$$p(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$$

such that

$$\int_0^1 p(x)x^k dx = 0$$

for $k = 0, 1, \dots, n - 1$?

Answer:

20. Five honest pirates stole a number of coconuts. Before going to sleep, they agreed to meet in the morning and divide the coconuts evenly between them. But during the night, one pirate woke up, took $1/5$ of the coconuts, and left; then another woke up, took $1/5$ of whatever was left (not knowing what the first pirate had done), and then the other three did likewise, one by one. Assuming that no pirate had to chop pieces off any coconut, what's the least

possible number of coconuts left over at the end, and with how many (at least) did they start?

Answer:

21. In both 2009 and 2010 team A had a higher goals per match ratio than team B ; is it possible that combining the two years, B had a higher games per match ratio than A !

Answer:

22. The minutes and hour hands on a clock are 4 and 3 units long, respectively. Find out the distance between the tips when the distance between them is changing most rapidly.

Answer:

23. Does there exist a continuous function f on $(0, \infty)$ to itself satisfying

$$f(f(x)) = x^x/2$$

for all $x > 0$?

Answer:

24. Suppose $P(x), Q(x) \in \mathbb{R}[x]$ are two non-constant polynomials such that

$$P(x)^n - 1 \mid Q(x)^n - 1$$

for all $n \in \mathbb{N}$. Does it follow that $Q(x)$ is a power of $P(x)k$ for some k

Answer:

25. Suppose the map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has the property that $f(K)$ is compact for all compact K , and $f(K)$ is connected for all connected K . Does it follow that f is continuous?

Answer:

26. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $\sum_i f(x_i)$ converges whenever $\sum_i x_i$ converges, then f is linear on a neighbourhood of 0.

Answer:

27. Suppose there is a 1000×1000 matrix without any entries filled in and two players A and B play a game where A and B take turns filling in entries in the matrix with A going first. If when all of the entries are filled in, the determinant of the matrix is 0, then B wins; otherwise A wins. What is the optimal strategy for B to use?

Answer: