Problem Solving (MA2201)

Week 6

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1. Does there exist a functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$f(f(n)) = n^2$$
for all n?

2. Does there exist an infinite uncountable family of subsets of $\mathbb{N}$ such that for all sets $A \neq B$ in this family either $A \subset B$ or $B \subset A$?

3. Let G be a group. Suppose that for all $x, y \in G$, x^2 and y^2 commute, and also x^3 and y^3 commute. Does it follow that G is an abelian group ?

4. If a is the sum of all the digits of $n = 2011^{2011}$, b is the sum of all the digits of a, and c is the sum of all the digits of b, find c.

5. What is the value of
$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}?$$

6. Find all primes p for which 2^p has last digit 4.

7. Let (x_n) be a sequence of real numbers satisfying
$$\lim_{n\to\infty}(x_n - x_{n-1}) = 0.$$
Prove that
$$\lim_{n\to\infty}\frac{x_n}{n} = 0.$$

8. Show that for each real number $\epsilon > 0$ there exist positive integers m, n such that
$$0 < \sqrt{n} - \sqrt{m} - \pi < \epsilon.$$

9. Given 11 integers $x_1, ..., x_{11}$ show that there must exist some non-zero finite sequence $a_1, ..., a_{11}$ of elements from $\{-1, 0, 1\}$ such that the sum $a_1x_1 + \cdots + a_{11}x_{11}$ is divisible by 2011.

10. Find all $c > 0$ for which the inequality
$$c^x \geq cx$$
holds for all $x > 0$.

11. If a is the sum of all the digits of $n = 2011^{2011}$, b is the sum of all the digits of a, and c is the sum of all the digits of b. Find c.

12. Show that the sequence
$$a_n = \sin(n^2)$$
is not convergent.

13. Suppose $p(x)$ is a polynomial with integer coefficients, and suppose $p(n)$ is a perfect square for all integers n. Show that $p(x) = q(x)^2$ for some polynomial $q(x)$.

14. A set of 2011 coins has the property that if any coin is removed, the remaining 2010 coins can be divided into two sets of 1005 coins having the same total weight. Show that all the coins must have the same weight.

15. Show that if the complex numbers z_1, z_2, z_3 satisfy the relation
$$2/z_1 = 1/z_2 + 1/z_3$$
then z_1, z_2, z_3 lie on a circle passing through the origin.

Challenge Problem

Suppose a rectangle can be divided into a finite number of squares. Prove that the ratio of the sides is rational.