Problem Solving (MA2201)

Week 6

Timothy Murphy

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1. Does there exist a functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) = n^2$$

for all n?

Answer: *Yes. To start with, let us set*

$$f(0) = 0, \ f(1) = 1.$$

Now for each number $n \neq 0, 1$ that is not a square let us form the sequence

$$S(n) = \{n, n^2, n^4, n^8, \dots\};$$

and let us pair off these subsets, $S(2)$ with $S(3)$. $S(5)$ with $S(6)$, etc.

If $S(m), S(n)$ are paired, let

$$m^{2^e} \mapsto n^{2^e}, \ n^{2^e} \mapsto n^{2^{e+1}}.$$

So, for example,

$$2 \mapsto 3, \ 2^2 \mapsto 3^2, \ 2^4 \mapsto 3^4, \dots 3 \mapsto 2^2, \ 3^2 \mapsto 2^4, \ 3^3 \mapsto 2^8, \dots$$

This gives a map with the required property.

2. Does there exist an infinite uncountable family of subsets of $\mathbb{N}$ such that for all sets $A \neq B$ in this family either $A \subset B$ or $B \subset A$?

Answer: *Yes. Since* $\mathbb{Q}$ *has the same cardinality as* $\mathbb{N}$ *(ie there is a bijection between them), we can consider subsets of* $\mathbb{Q}$ *in place of subsets of* $\mathbb{N}$.

But now we can identify each real number $\alpha \in \mathbb{R}$ *with the subset* $R_\alpha = \{q \in \mathbb{Q} : q > \alpha\}$ *(as in the 'Dedekind section' definition of the reals).*

Now

$$\alpha < \beta \implies R_\alpha \subset R_\beta;$$

and since the reals are totally ordered, so are the subsets R_α.

3. Let G be a group. Suppose that for all $x, y \in G$, x^2 and y^2 commute, and also x^3 and y^3 commute. Does it follow that G is an abelian group ?

4. If a is the sum of all the digits of $n = 2011^{2011}$, b is the sum of all the digits of a, and c is the sum of all the digits of b, find c.

   Answer:

5. What is the value of

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}?$$

Answer: *We have to find the limit of the sequence* (a_n) *given by*

$$a_{n+1} = \sqrt{2 + a_n},$$

with $a_0 = 0$.

If the sequence has a limit, say ℓ, *then*

$$\ell = \sqrt{2 + \ell}.$$

Thus

$$\ell^2 = 2 + \ell,$$

ie

$$\ell = \frac{1}{2}(1 \pm 9).$$

Since $\ell > 0$ it follows that

$$\ell = 5.$$

But does the sequence converge?

From above, $a_n < a_{n+1}$ if

$$a_n < \sqrt{2 + a_n},$$

ie

$$a_n^2 < 2 + a_n,$$

ie

6. Find all primes p for which 2^p has last digit 4.

   Answer: *Since*

   $$2^5 \equiv 2 \bmod 10,$$

   it follows that

   $$2^1, 2^2, 2^3, 2^4, 2^5, 2^6, \dots \equiv 2, 4, 8, 6, 2, 4, \dots \bmod 10$$

   Thus 2^n ends in 4 if and only if

   $$n \equiv 2 \bmod 4.$$

   In particular n must be even. Hence the only prime with this property is $p = 2$.

7. Let (x_n) be a sequence of real numbers satisfying

   $$\lim_{n\to\infty} (x_n - x_{n-1}) = 0.$$

   Prove that

   $$\lim_{n\to\infty} \frac{x_n}{n} = 0.$$

Answer: *Suppose* $\epsilon > 0$. *Then*

$$|x_n - x_{n-1}| < \epsilon$$

for $n \geq N$.

Now

$$x_n = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_{N+1} - x_N) + x_N.$$

Hence

$$|x_n| \leq (n - N)\epsilon + x_N,$$

and so

$$\left|\frac{x_n}{n}\right| \leq \epsilon + \left|\frac{x_N}{n}\right|.$$

But

$$\left|\frac{x_N}{n}\right| < \epsilon$$

for $n \geq N' \geq N$. *Thus*

$$\left|\frac{x_n}{n}\right| \leq 2\epsilon$$

for $n \geq N'$. *Hence*

$$\left|\frac{x_n}{n}\right| \to 0$$

as $n \to \infty$.

8. Show that for each real number $\epsilon > 0$ there exist positive integers m, n such that

$$0 < \sqrt{n} - \sqrt{m} - \pi < \epsilon.$$

Answer: *We have*

$$\sqrt{r+1} - \sqrt{r} = \frac{1}{\sqrt{r+1} + \sqrt{r}} < \epsilon$$

if

$$r \geq 1/\epsilon^2.$$

Thus if $m \geq 1/\epsilon^2$ *then each of*

$$\sqrt{m+1} - \sqrt{m}, \sqrt{m+2} - \sqrt{m+1}, \sqrt{m+3} - \sqrt{m+2}, \ldots$$

is $< \epsilon$*, ie the increasing numbers*

$$0, \sqrt{m+1} - \sqrt{m}, \sqrt{m+2} - \sqrt{m}, \sqrt{m+3} - \sqrt{m}, \ldots$$

are all $< \epsilon$ *apart.*

Since the numbers tend to ∞*, it follows that at least one falls in the interval* $(\pi, \pi + \epsilon)$*, say*

$$\pi < \sqrt{m+s} - \sqrt{m} < \pi + \epsilon.$$

Thus $n = m + s$ *provides a solution to the problem.*

9. Given 11 integers $x_1, ..., x_{11}$ show that there must exist some non-zero finite sequence $a_1, ..., a_{11}$ of elements from $\{-1, 0, 1\}$ such that the sum $a_1x_1 + \cdots + a_{11}x_{11}$ is divisible by 2011.

   Answer: *We note that* $2^{11} = 2048 > 2011$*, so we have 'room' to make 11 2-way choices.*

   Consider the 2^11 *numbers*

   $$c_1x_1 + \cdots + c_{11}x_{11}$$

   where $c_i \in \{0, 1\}$ *for* $i = 1, \ldots, 11$*.*

   Two of these numbers must be in the same residue class mod2011*, say*

   $$c_1x_1 + \cdots + c_{11}x_{11} \equiv d_1x_1 + \cdots + d_{11}x_{11} \bmod 2011;$$

   and then

   $$a_1x_1 + \cdots + a_{11}x_{11} \equiv 0 \bmod 2011$$

   on setting

   $$a_i = c_i - d_i \quad (1 \leq i \leq 11),$$

   noting that

   $$a_i \in \{-1, 0, 1\}$$

   for all i*.*

10. Find all $c > 0$ for which the inequality

$$c^x \geq cx$$

holds for all $x > 0$.

Answer: *If $x = 1$ then evidently $c^x = cx$.*

If not then

$$\begin{aligned} c^x \geq cx &\iff x \log c \geq logc + \log x \\ &\iff (x-1)\log c \geq \log x. \end{aligned}$$

If $x > 1$ this gives

$$\log c \geq \frac{\log x}{x-1}.$$

On the other hand, if $0 < x \leq 1$ then, on setting $y = 1/x$, we get

$$(1 - 1/y)\log c \leq \log y,$$

ie

$$\log c \leq \frac{y \log y}{y-1}$$

for all $y \geq 1$.

But as $x \to 1$ from above,

$$\frac{\log x}{x-1} \to 1,$$

by l'Hopital's rule.

Similarly, as $y \to 1$ from above,

$$\frac{y \log y}{y-1} \to 1.$$

Thus it is impossible for the inequality to hold for all x unless

$$\log c = 1,$$

ie

$$c = e.$$

It remains to determine if

$$e^x > ex$$

for all $x > 0$.

Let

$$f(t) = e^t - et.$$

Then

$$f'(t) = e^t - e.$$

Thus

$$f'(t) \begin{cases} < 0 \ \textit{if}\ 0 < t < 1 \\ = 0 \ \textit{if}\ t = 1 \\ > 0 \ \textit{if}\ t > 1 \end{cases}$$

It follows that $f(t)$ *decreases from* $f(0) = 1$ *as* t *increases from 0 to 1, and then increases from* $f(1) = 0$ *as* t *increases from 1. Hence* $f(t)$ *reaches its minimum at* $t = 1$*, and so*

$$e^t \geq et$$

for all $t > 0$.

Accordingly, the given inequality holds if and only if $c = e$.

11. If a is the sum of all the digits of $n = 2011^{2011}$, b is the sum of all the digits of a, and c is the sum of all the digits of b. Find c.

    Answer: *If* s *is the sum of digits of* r *then* $s \equiv r \bmod 9$. *Hence*

    $$a \equiv b \equiv c \bmod 9.$$

    Now

    $$2011 \equiv 4 \bmod 9,$$

and so

$$2011^{2011} \equiv 4^{2011} \bmod 9.$$

But

$$4^3 \equiv 1 \bmod 9,$$

and

$$2011 \equiv 1 \bmod 3.$$

Hence

$$2011^{2011} \equiv 4bmod9.$$

Thus

$$c \equiv 4 \bmod 9.$$

But how big is c? Since $2011^3 < 10^{10}$*, it follows that* $2011^{2011} < 10^{7000}$*. Thus* 2011^{2011} *has* < 7000 *digits, and so* $a < 7000 \cdot 9 < 100000$ *Thus* a *has* ≤ 5 *digits, of which the first is* ≤ 6*. Hence* $b \leq 6 + 4 \cdot 9 = 42$*, and so* $c \leq 12$*.*

It follows that

$$c = 4.$$

12. Show that the sequence

$$a_n = \sin(n^2)$$

is not convergent.

Answer: *It is sufficient to show that*

$$n^2 \bmod 2\pi$$

is dense in the interval $[0, 2\pi)$*, ie given any interval* (a, b) *there exist an infinity of* n *such that*

$$a < n^2 < b \bmod 2\pi.$$

13. Suppose $p(x)$ is a polynomial with integer coefficients, and suppose $p(n)$ is a perfect square for all integers n. Show that $p(x) = q(x)^2$ for some polynomial $q(x)$.

    Answer:

14. A set of 2011 coins has the property that if any coin is removed, the remaining 2010 coins can be divided into two sets of 1005 coins having the same total weight. Show that all the coins must have the same weight.

    Answer:

15. Show that if the complex numbers z_1, z_2, z_3 satisfy the relation
    $$2/z_1 = 1/z_2 + 1/z_3$$
    then z_1, z_2, z_3 lie on a circle passing through the origin.

    Answer:

Challenge Problem

Suppose a rectangle can be divided into a finite number of squares. Prove that the ratio of the sides is rational.