Problem Solving (MA2201)

Week 5

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1. Find the sum of the series
$$\sum_{n=1}^{\infty} \frac{n}{n^4+4}.$$

2. A convex polygon is drawn inside a square of side 1. Prove that the sum of the squares of the lengths of the sides of the polygon is at most 4.

3. Two lines m, n are given, and a positive number d. What is the locus of a point whose perpendicular distances from m and from n add up to d?

4. What is the last digit of the 100th number in the sequence
$$3,\ 3^3,\ 3^{3^3}, \ldots?$$

5. Solve the equation
$$(x-2)(x-3)(x+4)(x+5) = 44.$$

6. Show that if the integer n does not end in the digit 0 then one can find a multiple of n containing no 0's.

7. Show that in a group of 10 people there are either 3 people who know each other ("mututal acquaintances") or 4 people who don't know each other ("Mutual strangers").

8. Find all solutions in integers to the equation
$$x^2 + y^2 + z^2 = 2xyz.$$

9. How many ways are there of painting the 6 faces of a cube in 6 different colours, if two colourings are considered the same when one can be obtained from the other by rotating the cube?

10. How many positive integers $x < 2011$ are there such that 7 divides $2^x - x^2$?

11. If $x_1, x_2, \ldots, x_n$ are positive numbers with sum s, prove that
$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \leq 1 + s + \frac{s^2}{2!} + \cdots + \frac{s^n}{n!}.$$

12. Prove that in a finite group G the number of solutions of the equation $x^n = e$ is a multiple of n whenever n divides the order of the group.

13. There is a rabbit is in the middle of a circular pond. A poacher is on the edge of the pond. The poacher can run 4-times as fast as the rabbit can swim. Can the rabbit get away ?

14. Does there exist an infinite uncountable family of subsets of $\mathbb{N}$ such that $A \cap B$ is finite for all $A \neq B$ from this family?

15. For which real numbers $x > 0$ is there a real number $y > x$ such that
$$x^y = y^x \text{ ?}$$

Challenge Problem

Show that the equation
$$x^x y^y = z^z$$
has an infinity of solutions in integers $x, y, z > 1$.