Problem Solving (MA2201)

Week 5

Timothy Murphy

1. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{n^4+4}.$$

Answer: *Questions like this can sometimes be solved by using partial fractions, if they have a solution.*

In this case

$$n^4 + 4 = (n - \sqrt{2}\omega)(n - \sqrt{2}\omega^3)(n - \sqrt{2}\omega^5)(n - \sqrt{2}\omega^7)$$

where $\omega = e^{2\pi/8}$.

We can combine conjugate factors to give 2 real quadratics

$$x^4 + 4 = (x^2 + ax + b)(x^2 + cx + d).$$

From the coefficients of x^3 *and* x, *we must have* $c = -a,\ b = d$.

$$x^4 + 4 = (x^2 + ax + b)(x^2 - ax + b).$$

Thus

$$-a^2 + 2b = 0,\ b^2 = 4.$$

Hence $b = 2,\ a = 2$, *yielding*

$$x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2).$$

Now

$$\frac{1}{n^2 - 2n + 2} - \frac{1}{n^2 + 2n + 2} = \frac{4n}{n^4 + 4}$$

Hopefully, the 2 terms will cancel out in some way. In fact

$$(n+2)^2 - 2(n+2) + 2 = n^2 + 2n + 2.$$

Thus the first term with $n+2$ cancels out the second term with n. We are left with the first term for $n = 1, 2$.

$$\sum \frac{n}{n^4+4} = \frac{1}{4}\left(\frac{1}{1} + \frac{1}{2}\right) = \frac{3}{8}.$$

2. A convex polygon is drawn inside a square of side 1. Prove that the sum of the squares of the lengths of the sides of the polygon is at most 4.

   Answer:

3. Two lines m, n are given, and a positive number d. What is the locus of a point whose perpendicular distances from m and from n add up to d?

   Answer:

4. What is the last digit of the 100th number in the sequence

$$3,\ 3^3,\ 3^{3^3}, \ldots ?$$

   Answer: *We have the sequence*

$$a_{n+1} = 3^{a_n},$$

with $a_1 = 1$. We have to determine $a_{100} \bmod 10$.
Since

$$3^4 \equiv 1 \bmod 10,$$

it follows that

$$a_{n+1} \equiv \begin{cases} 1 \bmod 10 \ \textit{if}\ a_n \equiv 0 \bmod 4 \\ 3 \bmod 10 \ \textit{if}\ a_n \equiv 1 \bmod 4 \\ -1 \bmod 10 \ \textit{if}\ a_n \equiv 2 \bmod 4 \\ -3 \bmod 10 \ \textit{if}\ a_n \equiv 3 \bmod 4 \end{cases}$$

But

$$3^2 \equiv 1 \bmod 4,$$

it follows that

$$a_n \equiv \begin{cases} 1 \bmod 4 \text{ if } a_{n-1} \text{ is even} \\ 3 \bmod 4 \text{ if } a_{n-1} \text{ is odd} \end{cases}$$

Evidently a_n *is odd for all* n. *In particular* a_{98} *is odd. Hence*

$$a_{99} \equiv 1 \bmod 4,$$

and so

$$a_{100} \equiv 3 \bmod 10,$$

ie the last digit of a_{100} *is 3.*

5. Solve the equation

$$(x-2)(x-3)(x+4)(x+5) = 44.$$

Answer: *Writing* $x = t - 1$, *the equation becomes*

$$(t-3)(t-4)(t+3)(t+4) = 44,$$

ie

$$(t^2-9)(t^2-16) = 44.$$

Thus $u = t^2$ *satisfies*

$$(u-9)(u-16) = 44,$$

ie

$$u^2 - 25u + 100 = 0.$$

Writing $u = 5v$,

$$v^2 - 5v + 4 = 0,$$

ie

$$(v-1)(v-4)=0.$$

Thus

$$v = 1 \textit{ or } 4,$$

ie

$$u = 5 \textit{ or } 20.$$

Hence

$$t = \pm\sqrt{5} \textit{ or } 2 \pm 5,$$

and so

$$x = \sqrt{5}-1, -\sqrt{5}-1, 2\sqrt{5}-1 \textit{ or } -2\sqrt{5}-1.$$

6. Show that if the integer n does not end in the digit 0 then one can find a multiple of n containing no 0's.

   Answer: *Suppose first that* $\gcd(n, 10) = 1$.

   Let

$$N = 10^e - 1 = \overbrace{9\ldots9}^{e \text{ 9's}}.$$

   Then

$$n \mid N \iff 10^e \equiv 1 \bmod n,$$

   which holds if e is a multiple of the order of 10 in the group $(\mathbb{Z}/m)^\times$.

   Now suppose 2 or 5 divides n. The argument is the same in both cases, so we may assume that

$$n = 2^f m,$$

   where $\gcd(m, 10) = 1$.

   Note first thst a number is divisible by 2^e *if and only if the number formed by the last e digits is divisible by* 2^e *(since* $2^e \mid 10^f$ *if* $f \geq e$*).*

It is easy to find a number E with e non-zero digits divisible by 2^e. For if we have a multiple of 2^e with a zero r places from the end, and no zeros beyone that, then we can eliminate this zero by adding $10^{r-1}2^e$ (since the last digit of 2^e cannot be zero); and then we can eliminate any digits beyond the eth place.

Now we use the same argument as in the previous case, except that now we repeat E rather than 9. Thus we consider

$$\begin{aligned} N &= \overbrace{EE\ldots E}^{f\ E\text{'s}} \\ &= \left(10^{e(f-1)} + 10^{e(f-2)} + \cdots + 10^e + 1\right) E \\ &= \frac{10ef - 1}{10^e - 1} E. \end{aligned}$$

This is divisible by 2^e, since it ends in E; and it is divisible by m if

$$10^{ef} \equiv 1 \bmod (10^e - 1)m,$$

which will be the case if f is a multiple of the order of 10 in the group

$$(\mathbb{Z}/((10^e - 1)m))^\times.$$

7. Show that in a group of 10 people there are either 3 people who know each other ("mututal acquaintances") or 4 people who don't know each other ("Mutual strangers").

   Answer:

8. Find all solutions in integers to the equation

$$x^2 + y^2 + z^2 = 2xyz.$$

   Answer: *Since $x^2 + y^2 + z^2$ is even, one of x, y, z is even. We may assume that x is even.*

   Then $4 \mid 2xyz$ and so

$$y^2 + z^2 \equiv 0 \bmod 4.$$

It follows that y, z are both even. (If both are odd then $x^2+y^2 \equiv 2 \bmod 4$; If one is odd and one even then $x^2+y^2 \equiv 1 \bmod 4$.)

Let $x = 2X,\ y = 2Y,\ z = 2Z$. Then

$$X^2 + y^2 + Z^2 = 4XYZ.$$

By the same argument, X, Y, Z are all even. Let $X = 2X',\ Y = 2Y,\ Z = 2Z'$. Then

$$X'^2 + y'^2 + Z'^2 = 8X'Y'Z'.$$

Continuing in this way, $x, y.z$ are all divisible by an arbitrarily high power of 2, which is absurd.

9. How many ways are there of painting the 6 faces of a cube in 6 different colours, if two colourings are considered the same when one can be obtained from the other by rotating the cube?

   Answer: *Let us apply the Burnside-Polya Lemma.*

   The cube has 24 rotational symmetries: rotations through $\pm 2\pi/3$ about the 4 diagonals, giving 2 conjugacy classes, each containing 4 symmetries; rotations about the lines joining mid-points of opposite faces

10. How many positive integers $x < 2011$ are there such that 7 divides $2^x - x^2$?

    Answer: *Since*

    $$2^3 = 1 \bmod 7,$$

    It follows that

    $$2^x \equiv \begin{cases} 1 \bmod 7 \text{ if } x \equiv 0 \bmod 3 \\ 2 \bmod 7 \text{ if } x \equiv 1 \bmod 3 \\ 4 \bmod 7 \text{ if } x \equiv 2 \bmod 3 \end{cases}$$

On the other hand,

$$x^2 \equiv \begin{cases} 1 \bmod 7 \ \textit{if}\ x \equiv \pm 1 \bmod 7 \\ 2 \bmod 7 \ \textit{if}\ x \equiv \pm 3 \bmod 7 \\ 4 \bmod 7 \ \textit{if}\ x \equiv \pm 2 \bmod 7 \end{cases}$$

Thus

$$2^x \equiv x^2 \equiv 1 \bmod 7 \ \textit{if}\ x \equiv 15 \ \textit{or}\ 6 \bmod 21,$$
$$2^x \equiv x^2 \equiv 2 \bmod 7 \ \textit{if}\ x \equiv 10 \ \textit{or}\ 4 \bmod 21,$$
$$2^x \equiv x^2 \equiv 4 \bmod 7 \ \textit{if}\ x \equiv 2 \ \textit{or}\ 5 \bmod 21.$$

Since $21 \cdot 95 = 1995$*, there are* $6 \times 95 = 570$ *solutions in* $[1, 1995]$*. Since* $2010 \equiv 15 \bmod 21$*, there are 5 solutions in* $[1996, 2010]$.

Hence there are 575 solutions in all.

11. If x_1, x_2, $\dots$, x_n are positive numbers with sum s, prove that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \le 1 + s + \frac{s^2}{2!} + \cdots + \frac{s^n}{n!}.$$

Answer:

Method 1 *We have*

$$1 + x < e^x = 1 + x + x^2/2! + \cdots$$

Thus

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) < e^s = 1 + s + s^2/2! + \cdots$$

But if we consider this as an inequality in the variables $x_1, \dots, x_n$*, we observe that the terms on the left will only occur in the terms on the right up to* $s^n/n!$.
It follows that

$$(1+x_1)(1+x_2) \cdots (1+x_n) < e^s = 1+s+s^2/2!+\cdots+s^n/n!.$$

Method 2 *For given s, let us try to find the maximum of*

$$f(x_1, \ldots, x_n) = (1 + x_1) \cdots (1 + x_n)$$

subject to the constraint

$$x_1 + \cdots + x_n = s.$$

Applying the Lagrange multiplier method, at a stationary point

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_n}.$$

12. Prove that in a finite group G the number of solutions of the equation $x^n = e$ is a multiple of n whenever n divides the order of the group.

    Answer:

13. There is a rabbit is in the middle of a circular pond. A poacher is on the edge of the pond. The poacher can run 4-times as fast as the rabbit can swim. Can the rabbit get away ?

    Answer: *The rabbit can escape.*

    Suppose the pond has radius R, and the rabbit can swim at speed v.

    Let us say the poacher starts at angle π at the edge of the pond, and starts to run round the edge in anti-clockwise direction at speed v.

    And let us suppose that the rabbit starts swimming along the radius at angle α.

    To reach the rabbit the poacher has to run through angle $\pi + \alpha$. This will take time $R(\pi + \alpha)/4v$.

    Meanwhile the rabbit takes time R/v to reach the edge.

    Thus the rabbit will reach the edge before the poacher gets there if

$$(\pi + \alpha)/4 > 1,$$

ie

$$\alpha > \pi - 4.$$

So the rabbit aims for a point a little larger than $\pi - 4$.

Suppose the poacher is clever, and reverses direction after going through angle θ, *in time* $R\theta/4v$.

The rabbit now aims for position $-\beta$.

The poacher has to go through angle $\pi + 2\theta + \beta$.

Meanwhile the rabbit is $R\cos\alpha$ *nearer the edge, but has to travel through angle* $\alpha + \beta$.

14. Does there exist an infinite uncountable family of subsets of $\mathbb{N}$ such that $A \cap B$ is finite for all $A \neq B$ from this family?

    Answer:

15. For which real numbers $x > 0$ is there a real number $y > x$ such that

$$x^y = y^x \ ?$$

Answer: *If* $x^y = y^x$ *(and* $x, y > 0$*) then*

$$x^{1/x} = y^{1/y}.$$

Consider the function

$$f(x) = x^{1/x} = e^{\log x/x}$$

for $x > 0$. *We have to find for which* c *the equation*

$$f(x) = c$$

has 2 (or more) solutions.

We have

$$f'(x) = (1/x^2 - \log x/x^2)e^{\log x/x}.$$

Thus

$$f'(x) = 0 \iff x = e.$$

As $x \to 0$ *(from above),* $\log x/x \to -\infty$ *and so* $f(x) \to 0$*.*

As $x \to \infty$*,* $\log x/x \to 0$ *and so* $f(x) \to 1$*.*

We see that $f(x)$ *increases from 0 (at* $x = 0$*) to a maximum of* $e^{1/e}$ *at* $x = e$*, and then decreases to 1 at* ∞*.*

It follows that the equation

$$f(x) = c$$

has 2 solutions if and only if

$$1 < c;$$

and the smallest of these solutions will satisfy

$$0 < x < e.$$

Challenge Problem

Show that the equation

$$x^x y^y = z^z$$

has an infinity of solutions in integers $x, y, z > 1$.

Answer: