Problem Solving (MA2201)

Week 4

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1. Evaluate
$$\sum_{j=2}^{\infty}\left(\sum_{i=2}^{\infty}\frac{1}{i^j}\right).$$

2. The point P lies inside the square $ABCD$. If $|AP| = |BP|$ and $A\hat{B}P = 15°$, show that the triangle CPD is equilateral.

3. Construct an infinite non-constant arithmetic sequence of positive integers which contains no squares, cubes or higher powers of integers.

4. Show that the number of ways of making up n cents from 1c, 2c and 5c pieces is the nearest integer to $\dfrac{(n+4)^2}{20}$.

5. For which positive real numbers x does the sequence
$$x,\ x^x,\ x^{x^x},\ \ldots$$
converge?

6. Show that if a pentagon $ABCDE$ inscribed in a circle has equal angles then it has equal sides.

7. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that
$$f(f(n)) = n + 1$$
for all n?

8. Let $p(x)$ be the polynomial of degree n such that

$$p(k) = \frac{k}{k+1}, \qquad k = 0, 1, 2, \ldots, n.$$

Find $p(n+1)$.

9. Does the number 2011^n end in the digits 2011 for any integer $n > 1$?

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11. If A, B, C are the angles of a triangle, what is the maximal value of

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

and when is it attained?

12. S is a sphere with centre O. Given any point P outside S let S' be the sphere with centre P passing through O. Show that the area of that part of the surface of S' lying inside S is independent of P.

13. A piece is broken off each of 3 equal rods at random. What is the probability that these 3 pieces form a triangle?

14. The points $P_1, \ldots, P_n$ lie on the surface of a sphere of radius r. Show that the distance between the 2 closest points is $< 4r/\sqrt{n}$.

15. Suppose A and B are two different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, show that $A^2 + B^2$ cannot be invertible.

Challenge Problem

Alice and Bob play the following game on an infinite square grid: taking turns, they paint the sides of the squares, Alice in red and Bob in blue. The same side cannot be painted twice. Alice plays first. Show that Bob can prevent Alice completing a closed red contour.