Problem Solving (MA2201)

Week 4

Timothy Murphy

1. Evaluate

$$\sum_{j=2}^{\infty}\left(\sum_{i=2}^{\infty}\frac{1}{i^j}\right).$$

Answer: *Since the terms are all positive, we can reverse the order of summation:*

$$\begin{aligned} S &= \sum_{i=2}^{\infty}\left(\sum_{j=2}^{\infty}\frac{1}{i^j}\right) \\ &= \sum_{i=2}^{\infty}\left(\frac{1}{i^2}+\frac{1}{i^3}+\frac{1}{i^4}+\cdots\right) \\ &= \sum_{i=2}^{\infty}\left(\frac{1/i^2}{1-1/i}\right) \\ &= \sum_{i=2}^{\infty}\left(\frac{1}{i(i-1)}\right) \end{aligned}$$

We know that

$$\sum_{i=2}^{\infty} x^{i+1} = \frac{1}{1-x} - (1+x+x^2).$$

Integrating from 0 to t,

$$\sum_{i=2}^{\infty}\frac{t^i}{i} = -\log(1-t) - (t + t^2/2 + t^3/3).$$

Integrating this from 0 to x,

$$\sum_{i=2}^{\infty} \frac{x^{i-1}}{i(i-1)} = ((1-x)\log(1-x))+x)-(x^2/2+x^3/6+x^4/12)$$

Setting $x = 1$*,*

$$\begin{aligned} S &= \sum_{i=2}^{\infty} \frac{1}{i(i-1)} \\ &= 1 - (1/2 + 1/6 + 1/12) \\ &= 1/4, \end{aligned}$$

2. The point P lies inside the square $ABCD$. If $|AP| = |BP|$ and $A\hat{B}P = 15^\circ$, show that the triangle CPD is equilateral.

   Answer: *Let us draw an equilateral triangle* CPD *(pointing into the square). Then*

   $$B\hat{C}P = \pi/2 - P\hat{C}D = \pi/2 - \pi/3 = \pi/6.$$

   Let

   $$A\hat{B}P = x.$$

   Then

   $$C\hat{B}P = \pi/2 - x.$$

   Now

   $$CP = CD = CB.$$

   Thus the triangle CPB *is isosceles. Hence*

   $$C\hat{B}P = C\hat{P}B = \pi/2 - x.$$

   So from the triangle BCP*,*

   $$2(\pi/2 - x) + \pi/6 = \pi,$$

   ie

   $$A\hat{B}P = x = \pi/12 = 15^\circ.$$

   Conversely, if we start with $A\hat{B}P = 15^\circ$ *then we obtain an equilateral triangle* CPD*.*

3. Construct an infinite non-constant arithmetic sequence of positive integers which contains no squares, cubes or higher powers of integers.

   Answer: *Let p be a prime. Consider the arithmetic sequence*

   $$a(n) = p + np^2$$

   for $n = 1, 2, \ldots$.

   Since $p \mid a(n)$ for all n, if $a(n)$ is a square or higher power then

   $$p^2 \mid a(n),$$

   which is not the case since

   $$a(n) \equiv p \bmod p^2.$$

4. Show that the number of ways of making up n cents from 1c, 2c and 5c pieces is the nearest integer to $\dfrac{(n+4)^2}{20}$.

   Answer: *Although it is possible to solve this question using generating series, it turns out to be much easier to prove it using induction.*

   Let the number of ways of making n cents from 1c, 2c and 5c coins be $a(n)$; and let the number of ways of making n cents just from 1c and 2c coins be $b(n)$. Evidently

   $$b(n) = \lfloor n/2 \rfloor + 1.$$

   We can divide the ways of making up n cents in two; those that only use 1c and 2c coins, and those that use at least one 5c coin.

   In the second case, if we remove one 5c coin there are $a(n-5)$ ways of making up the remaining sum.

   Thus

   $$a(n) = a(n-5) + b(n).$$

   Applying the same argument again,

   $$\begin{aligned} a(n) &= a(n-10) + b(n) + b(n-5) \\ &= a(n-10) + \lfloor n/2 \rfloor + 1 + \lfloor (n-5)/2 \rfloor + 1. \end{aligned}$$

If n is even then $n-5$ is odd, and

$$\lfloor n/2 \rfloor + \lfloor (n-5)/2 \rfloor = n/2 + (n-6)/2 = n-3.$$

Similarly, if n is odd then $n-5$ is even, and

$$\lfloor n/2 \rfloor + \lfloor (n-5)/2 \rfloor = (n-1)/2 + (n-5)/2 = n-3,$$

as before. Thus in all cases

$$a(n) = a(n-10) + (n-1).$$

Replacing n by $n+10$,

$$a(n+10) = a(n) + (n+9).$$

Now let

$$f(n) = (n+4)^2/20 = (n^2 + 8n + 16)/20.$$

We have

$$\begin{aligned} f(n+10) &= (n+4+10)^2/20 \\ &= (n+4)^2/20 + 20(n+4)/20 + 100/20 \\ &= f(n) + n + 4 + 5 \\ &= f(n) + 9. \end{aligned}$$

Thus the fractional parts of $f(n)$ and $f(n+10)$ are the same; and the closest integers differ by 9, like $a(n)$ and $a(n+10)$.

It follows that if the result holds for some n (ie $a(n)$ is equal to the nearest integer to $f(n)$) then the same will be true for $n+10, n+20, \ldots$.

Thus to prove the result for all n, it is sufficient to verify it for $1 \le n \le 10$.

We summarise this in a table:

n	$a(n)$	$(n+4)^2/20$
1	1	1.25
2	2	1.8
3	2	2.45
4	3	3.2
5	4	4.05
6	5	5.0
7	6	6.05
8	7	7.2
9	8	8.45
10	10	9.8

We see that in all these cases $a(n)$ is indeed the nearest integer to $f(n)$. Hence the result holds for all $n \geq 1$.

5. For which positive real numbers x does the sequence

$$x,\ x^x,\ x^{x^x},\ \ldots$$

converge?

Answer: *We are dealing with the sequence $a(0), a(1), \ldots$ defined by*

$$a(n+1) = x^{a(n)},$$

with $a(0) = 1$.

Suppose

$$a(n) \to \ell$$

as $n \to \infty$. Then

$$a(n+1) \to x^\ell.$$

Hence

$$x^\ell = \ell,$$

ie

$$x = \ell^{1/\ell}.$$

Consider the function

$$f(t) = t^{1/t}$$

for $t > 0$.

We have

$$\log f(t) = \frac{\log t}{t},$$

and so

$$f'(t) = \frac{1 - \log t}{t^2}.$$

Thus

$$f'(t) = 0$$

when $t = e$ *and*

$$f(e) = e^{1/e}.$$

Since

$$\log f(t) \to -\infty$$

as $t \to 0$ *(from above), it follows that*

$$f(t) \to 0 \text{ as } t \to 0$$

On the other hand,

$$\log f(t) \to 0$$

as $t \to \infty$, *and so*

$$f(t) \to 1 \text{ as } t \to \infty.$$

Hence $f(t)$ *increeases from 0 to* $e^{1/e}$ *as* t *increases from 0 (or near 0) to* e *and then decreases to 1 as* t *increase to* ∞.

It follows that $a(n)$ *is not convergent if* $x > e^{1/e}$.

If $0 < x < 1$ *then*

$$a < 1 \implies a^x < a.$$

It follows that $a(n)$ is decreasing, and so converges.

If $x = 1$ then $a(n) = 1$ for all n, and so $a(n) \to 1$.

If $x > 1$ then

$$a > 1 \implies a^x > a.$$

Hence $a(n)$ is increasing, and so converges or tends to ∞.

We know that $a(n)$ diverges if $x > e^{1/e}$. It remains to determine the outcome if $1 < x \le \ell = e^{1/e}$.

If $1 < a \le \ell$ and $x \le \ell$ then

$$a^x \le \ell^\ell = \ell.$$

Hence $a(n)$ is bounded above, and so converges.

Thus $a(n)$ converges if and only if $0 < x \le e^{1/e}$.

6. Show that if a pentagon $ABCDE$ inscribed in a circle has equal angles then it has equal sides.

   Answer: *Let O be the centre of the circle; and let*

$$O\hat{A}B = x.$$

Then

$$O\hat{B}A = x,$$

since OAB is isosceles.

The sum of the angles in the pentagon is $5\pi - 2\pi = 3\pi$. Thus each of the angles is $3\pi/5$. In particular

$$A\hat{B}C = \frac{3}{5}\pi.$$

Hence

$$O\hat{B}C = A\hat{B}C - O\hat{B}A = \frac{3}{5}\pi - x.$$

7. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) = n + 1$$

for all n?

Answer: *The answer is No. For such a map f must commute with*

$$g(n) = n + 1,$$

since

$$f \circ g = f \circ (f \circ g) = (f \circ f) \circ f = g \circ f.$$

But it is easy to see that the only maps $h : \mathbb{Z} \to \mathbb{Z}$ which commute with g are the powers of g, ie

$$h(n) = g^r(n) = n + r$$

for $r \in \mathbb{Z}$. For if

$$h(0) = r$$

then

$$\begin{aligned} h(1) &= h(g(0)) = g(h(0)) = h(0) + 1 = r + 1, \\ h(2) &= h(g(1)) = g(h(1)) = h(1) + 1 = r + 2, \end{aligned}$$

and so on, while

$$g(h(-1)) = h(g(-1)) = h(0) = r,$$

id

$$h(-1) + 1 = r \implies h(-1) = r - 1,$$

and so on.

Thus

$$f(n) = n + r$$

for some r, and so

$$f(f(n)) = n + 2r,$$

which cannot be $g(n) = n + 1$.

8. Let $p(x)$ be the polynomial of degree n such that

$$p(k) = \frac{k}{k+1}, \qquad k = 0, 1, 2, \ldots, n.$$

Find $p(n+1)$.

Answer: *Let*

$$f(x) = 1 - p(x).$$

Then

$$f(k) = \frac{1}{k+1}, \qquad k = 0, 1, 2, \ldots, n.$$

Now let

$$g(x) = f(x+1)$$

Then

$$g(k) = \frac{1}{k}, \qquad k = 1, 2, 3, \ldots, n+1.$$

Let

$$h(x) = xg(x)$$

Then

$$h(k) = 1, \qquad k = 1, 2, 3, \ldots, n+1.$$

Also,

$$h(0) = 0.$$

Thus

$$h(x) - 1$$

is a polynomial of degree $n+1$ *with zeros at* $1, 2, 3, \ldots, n+1$. *It follows that*

$$h(x) - 1 = c(x-1)(x-2)\cdots(x-(n+1))$$

for some constant c. *Since* $h(0) = 0$,

$$c = (-1)^n/(n+1)!.$$

Thus

$$h(x) = 1 + (-1)^n(x-1)(x-2)\cdots(x-(n+1))/(n+1)!.$$

Hence

$$\begin{aligned} h(n+2) &= 1 + (-1)^n \\ &= \begin{cases} 2 \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases} \end{aligned}$$

So

$$f(n+1) = g(n+2) = \begin{cases} 2/(n+2) \textit{ if } n \textit{ is even} \\ 0 \textit{ if } n \textit{ is odd} \end{cases}$$

and finally

$$p(n+1) = 1 - f(n+1) = \begin{cases} n/(n+2) \textit{ if } n \textit{ is even} \\ 1 \textit{ if } n \textit{ is odd} \end{cases}$$

9. Does the number 2011^n end in the digits 2011 for any integer $n > 1$?

   Answer: *Yes.*

   Since 2011 is coprime to 10^4, *2011 is an element of the finite group* $(\mathbb{Z}/10^4)^\times$, *and so has finite order* d:

   $$2011^d \equiv 1 \bmod 10000,$$

   ie 2011^d *ends in 10001. Hence*

   $$2011^{d+1} \equiv 2011 \bmod 10000,$$

   ie 2011^{d+1} *ends in the digits 2011.*

10. Does the number 2011^n start with the digits 2011 for any integer $n > 1$?

    Answer: *Yes.*

    2011^n *begins with* 2011 *if*

    $$2011 \cdot 10^m \le 2011^n < 2012 \cdot 10^m$$

    for some natural number m. *In other words,*

    $$10^m \le 2011^{n-1} < \frac{2012}{2011} 10^m.$$

    Taking logarithms to base 10, this is equivalent to

    $$m \le (n-1)\log 2011 < m + \log 2012/2011.$$

    Let $\{x\}$ *denote the 'fractional part' of* x, *ie*

    $$\{x\} = x - [x].$$

We have to show that some integral multiple of $\log 2011$ *has small fractional part.*

This is a general result: given any $\alpha \in \mathbb{R}$ *and any* $\epsilon > 0$ *we can find an* $n \in \mathbb{N}$ *such that*

$$\{n\alpha\} < \epsilon.$$

This is proved using the Pigeon-hole Principle. Choose N *so that*

$$N\epsilon > 1,$$

and consider the $N + 1$ *fractional parts*

$$0, \{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots, \{N\alpha\}.$$

11. If A, B, C are the angles of a triangle, what is the maximal value of

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

and when is it attained?

Answer: *Since the function is continuous, its maximum is attained at some point inside the triangle.*

Keeping C *fixed,*

$$\begin{aligned} \sin(A/2)\sin(B/2) &= \cos(A/2 - B/2) - \cos(A/2 + B/2) \\ &= \cos(A/2 - B/2) - \sin(C/2), \end{aligned}$$

since $(A + B)/2 = \pi/2 - C/2$.

Thus at a maximum

$$A = B.$$

Similarly

$$A = C.$$

Thus the maximum is attained where

$$A = B = C = \pi/3.$$

Since

$$\sin(\pi/6) = 1/2$$

the maximal value is $1/8$.

12. S is a sphere with centre O. Given any point P outside S let S' be the sphere with centre P passing through O. Show that the area of that part of the surface of S' lying inside S is independent of P.

Answer: *We use the theorem, due to Archimedes, that the surface area of the slice of a sphere cut off by a plane is equal to the corresponding area of the enclosing cylinder.*

Take a plane through OP, and let the circle $S \cap S'$ intersect this plane in Q, R. If QR meets OP in T then the theorem tells us that the surface area of S' inside S is

$$A = 2\pi OP \cdot OT.$$

We have to show that this does not depend on P.

The two right-angled triangles OQT and OPQ are similar. Hence

$$\frac{OT}{OQ} = \frac{OQ}{OP}.$$

Thus

$$OP \cdot OT = OQ^2 = r^2,$$

where r is the radius of S. In particular, $OP \cdot OT$ is independent of P.

13. A piece is broken off each of 3 equal rods at random. What is the probability that these 3 pieces form a triangle?

Answer: *We can suppose the rods have length 1 (since this is just a matter of choosing a unit of length). Suppose pieces of lengths x, y, z are broken off the 3 rods. These will form a triangle if and only if*

$$x \le y + z,\ y \le x + z,\ z \le x + y.$$

The values of x, y, z are distributed evenly over the unit cube I^3. Thus the required probability is just the volume of the region

$$S = \{(x, y, z) \in I^3 : x \le y + z, y \le x + z, z \le x + y\}.$$

Since

$$I^3 \setminus S = S_x \cup S_y \cup S_z,$$

where S_x, S_y, S_z *are the 3 disjoint sets*

$$S_x = \{(x, y, z) \in I^3 : x > y+z\},\ S_y = \{(x, y, z) \in I^3 : y > x+z\},\ S_z = \{(x, y$$

the probability is

$$1 - 3\|S_x\|.$$

Now S_x *is the tetrahedron with vertices*

$$(0, 0, 0),\ (1, 1, 0),\ (1, 0, 1),\ (1, 1, 1).$$

The first 3 of these lie in the plane $x = y + z$*, forming an equilateral triangle with side* $\sqrt{2}$*. The 4th point is distance*

$$\frac{|1 - 1 - 1|}{\sqrt{3}}$$

from the plane. Thus the volume of S_x *is*

$$\frac{1}{3}\frac{1}{2}\sqrt{2}\frac{\sqrt{3}}{\sqrt{2}}\frac{1}{\sqrt{3}} = \frac{1}{6}.$$

It follows that the probability of the pieces forming a triangle is

$$1 - 3\frac{1}{6} = \frac{1}{2}.$$

14. The points $P_1, \ldots, P_n$ lie on the surface of a sphere of radius r. Show that the distance between the 2 closest points is $< 4r/\sqrt{n}$.

Answer: *Suppose the smallest distance between 2 points is* d*. Let* α *the angle for which*

$$r \sin\alpha = \frac{d}{2}.$$

Draw a circle radius $d/2$ *around each of the points* P_i*, ie the circle formed by the points* P *such that* PP_i *subtends angle* α *at the centre of the sphere.*

Then these circles—or rather the areas on the surface of the sphere inside these circles—will not overlap. For if the circles around P_i and P_j overlap then

$$P_iOP_j \leq 2\alpha,$$

from which it follows that

$$|P_iP_j| \leq 2r\sin\alpha = d.$$

The area of each circle is

$$\pi(r\sin\alpha)^2,$$

so the area inside each circle on the sphere is larger than this. Since the circles do not overlap, their total area is less than the area of the sphere, ie

$$n\pi(r\sin\alpha)^2 \leq 4\pi r^2.$$

Thus

$$n(d/2)^2 \leq 4r^2,$$

from which the result follows.

15. Suppose A and B are two different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, show that $A^2 + B^2$ cannot be invertible.

    Answer: *Since*

    $$(A^2 + B^2)(A - B) = (A^3 - B^3) - (A^2B - B^2A) = 0,$$

    while $A - B \neq 0$, it follows that $A^2 + B^2$ is singular. (If v is a non-zero column of $A - B$ then $(A^2 + B^2)v = 0$.)

Challenge Problem

Alice and Bob play the following game on an infinite square grid: taking turns, they paint the sides of the squares, Alice in red and Bob in blue. The same side cannot be painted twice. Alice plays first. Show that Bob can prevent Alice completing a closed red contour.

Answer: