Problem Solving (MA2201)

Week 3

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1. Show that the product of any n successive integers is divisible by $n!$.

2. A rod of length 1 is thrown at random onto a floor tiled in squares of side 1. What is the probability that the rod will fall wholly within one square?

3. What point P in a triangle ABC minimises
$$AP + BP + CP?$$

4. Evaluate
$$\sum_{n=1}^{\infty} \frac{n^3}{3^n}.$$

5. What is the minimum value of
$$f(x) = x^x$$
for $x > 0$?

6. Prove that 3, 5 and 7 are the only 3 consecutive odd numbers all of which are prime.

7. In how many ways can $1,000,000$ be expressed as the product of 3 positive integers. (Factorisations differing only in order are to be considered the same.)

8. Prove that 2^n can begin with any sequence of digits.

9. The point P lies inside the square $ABCD$. If $|PA| = 5$, $|PB| = 3$ and $|PC| = 7$, what is the side of the square?

10. The function $f(x)$ satisfies $f(1) = 1$ and
$$f'(x) = \frac{1}{x^2 + f^2(x)}$$
for $x > 1$. Prove that
$$\lim_{x \to \infty} f(x)$$
exists and is less than $1 + \pi/4$.

11. Find the maximum value of
$$\frac{x + 2}{2x^2 + 3x + 6}.$$

12. Let $a_1, a_2, a_3, \ldots$ be the sequence of all positive integers with no 9's in their decimal representation. Show that
$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$
converges.

13. How many zeros does the function
$$f(x) = 2^x - 1 - x^2$$
have on the real line?

14. Suppose $f(x)$ is a polynomial with integer coefficients. If
$$f(f(f(f(n)))) = n$$
for some integer n, show that
$$f(f(n)) = n.$$

15. Show that for any positive reals a, b, c,
$$[(a + b)(b + c)(c + a)]^{1/3} \geq \frac{2}{\sqrt{3}}(ab + bc + ca)^{1/2}.$$

Challenge Problem

Suppose a, b are two positive integers such that $ab + 1$ divides $a^2 + b^2$. Prove that $(a^2 + b^2)/(ab + 1)$ is a perfect square.