Problem Solving (MA2201)

Week 3

Timothy Murphy

1. Show that the product of any n successive integers is divisible by $n!$.

2. A rod of length 1 is thrown at random onto a floor tiled in squares of side 1. What is the probability that the rod will fall wholly within one square?

   Answer:

3. What point P in a triangle ABC minimises
$$AP + BP + CP?$$

   Answer:

4. Evaluate
$$\sum_{n=1}^{\infty} \frac{n^3}{3^n}.$$

   Answer: *Let*
$$\begin{aligned} f(x) &= 1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots \\ &= \sum_0^{\infty} \frac{x^n}{3^n}. \end{aligned}$$

   (The series is convergent if $|x| < 3$.)

   Differentiating,
$$f'(x) = \sum_1^{\infty} \frac{nx^{n-1}}{3^n}$$

Hence

$$\begin{aligned} xf'(x) &= \sum_1^\infty \frac{nx^n}{3^n} \\ &= \sum_0^\infty \frac{nx^n}{3^n}. \end{aligned}$$

Differentiating again,

$$xf''(x) + f'(x) = \sum_1^\infty \frac{n^2 x^{n-1}}{3^n},$$

and so, multiplying by x again,

$$x^2 f''(x) + xf'(x) = \sum_0^\infty \frac{n^2 x^n}{3^n}.$$

Differentiating again,

$$x^2 f'''(x) + 3xf''(x) + f'(x) = \sum_1^\infty \frac{n^3 x^{n-1}}{3^n}.$$

Setting $x = 1$,

$$\Sigma = \sum_0^\infty \frac{n^3}{3^n} = f'''(1) + 3f''(1) + f'(1).$$

Now

$$\begin{aligned} f(x) &= \frac{1}{1 - x/3} \\ &= \frac{3}{3 - x}. \end{aligned}$$

Hence

$$\begin{aligned} f'(x) &= \frac{3}{(3-x)^2}, \\ f''(x) &= \frac{6}{(3-x)^3}, \\ f'''(x) &= \frac{18}{(3-x)^4}. \end{aligned}$$

Thus

$$f'(1) = \frac{3}{4},$$
$$f''(1) = \frac{6}{8} = \frac{3}{4},$$
$$f'''(1) = \frac{18}{16} = \frac{9}{8}.$$

It follows that

$$\begin{aligned}\Sigma &= \frac{9}{8} + 3\frac{3}{4} + \frac{3}{4} \\ &= \frac{17}{8}.\end{aligned}$$

5. What is the minimum value of

$$f(x) = x^x$$

for $x > 0$?

Answer: *We have*

$$\log f(x) = x \log x,$$

and so

$$\frac{f'(x)}{f(x)} = \log x + 1.$$

Hence

$$f'(x) = 0 \iff x = 1/e.$$

If $0 < x < 1/e$ *then* $\log x + 1 < 0$ *and so* $f(x)$ *is decreasing. If* $x > 1/e$ *then* $\log x + 1 > 0$ *and so* $f(x)$ *is increasing. Hence* $f(x)$ *attains its minimum value at* $x = 1/e$*, where*

$$f(1/e) = e^{-1/e}.$$

6. Prove that 3, 5 and 7 are the only 3 consecutive odd numbers all of which are prime.

Answer: *One of* $n, n+2, n+4$ *is divisible by 3, and if* $n > 3$ *this number is not prime.*

7. In how many ways can $1,000,000$ be expressed as the product of 3 positive integers. (Factorisations differing only in order are to be considered the same.)

   Answer: *We have*

$$1,000,000 = 2^6 5^6.$$

   The three factors must be

$$2^{e_1} 5^{f_1}, 2^{e_2} 5^{f_2}, 2^{e_3} 5^{f_3},$$

   where

$$0 \leq e_1, e_2, e_3 \leq 6$$

   and

$$e_1 + e_2 + e_3 = 6, f_1 + f_2 + f_3 = 6.$$

8. Prove that 2^n can begin with any sequence of digits.

   Answer:

9. The point P lies inside the square $ABCD$. If $|PA| = 5$, $|PB| = 3$ and $|PC| = 7$, what is the side of the square?

   Answer:

10. The function $f(x)$ satisfies $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f^2(x)}$$

   for $x > 1$. Prove that

$$\lim_{x \to \infty} f(x)$$

   exists and is less than $1 + \pi/4$.

   Answer:

11. Find the maximum value of

$$\frac{x+2}{2x^2 + 3x + 6}.$$

   Answer:

12. Let $a_1, a_2, a_3, \ldots$ be the sequence of all positive integers with no 9's in their decimal representation. Show that

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

   converges.

   Answer:

13. How many zeros does the function

$$f(x) = 2^x - 1 - x^2$$

have on the real line?

Answer: *Observe that*

$$f(0) = 0, \; f(1) = 0.$$

Also

$$f(2) = -1, \; f(3) = -2, \; f(4) = -1, \; f(5) = 6.$$

Thus $f(x)$ *has at least one zero* $\theta \in (4, 5)$.

We shall show that $0, 1, \theta$ *are the only zeros of* $f(x)$.

If $x < 0$ *then* $2^x < 1$ *and so* $f(x) < 0$. *Thus there are no negative zeros, and so we may assume that* $x \geq 0$.

We have

$$f'(x) = \log 2 \cdot 2^x - 2x$$

(since $2^x = e^{(\log 2)x}$*). Thus*

$$f''(x) = (\log 2)^2 2^x - 2.$$

Since 2^x *is increasing,* $f''(x)$ *has at most one zero. Since* $f''(1) < 0$ *(as* $\log 2 < 1$*), while* $f''(x) \to \infty$ *as* $x \to \infty$*, it follows that* $f''(x)$ *has just one zero, and this is* > 1.

If $f(x)$ *had a zero* $\phi \in (0, 1)$ *then, by the Mean Value Theorem,* $f'(x)$ *would have zeros in* $(0, \phi)$ *and in* $(\phi, 1)$*, and so* $f''(x)$ *would have a zero in* $(0, 1)$*, which we have seen is not the case. So* $f(x)$ *has no zeros in* $(0, 1)$.

Suppose $f(x)$ *has more than one zero in* $(1, \infty)$. *Let* α *be the smallest such zero, and* β *be the largest.*

Since $f'(1) < 0$*, while* $f'(x) > 0$ *for large* x*, it follows that*

Note that $f'(x)$ *does not have a multiple zero, since*

$$\begin{aligned} f'(x) = f''(x) = 0 &\implies (log2)f'(x) - f''(x) = 0 \\ &\implies (\log 2)x = 1 \\ &\implies 2^x = e. \end{aligned}$$

14. Suppose $f(x)$ is a polynomial with integer coefficients. If

$$f(f(f(f(n)))) = n$$

for some integer n, show that

$$f(f(n)) = n.$$

Answer: *If $u, v \in \mathbb{Z}$ then*

$$(u - v) \mid f(u) - f(v).$$

Suppose

$$f(n) = a, f(a) = b, f(b) = c, f(c) = n.$$

Then

$$n - a \mid f(n) - f(a),$$

ie

$$n - a \mid a - b.$$

Similarly

$$a - b \mid b - c,\ b - c \mid c - n\ n - a \mid a - b.$$

Hence

$$n - a = \pm(a - b) = \pm(b - c) = \pm(c - n).$$

If

$$n - a = -(a - b)$$

then

$$n = b,$$

which is what we have to prove.
Similarly

$$b - c = -(c - n) \implies b = n,$$

while

$$\begin{aligned} a - b = -(b - c) &\implies a = c \\ &\implies f(a) = f(c) \\ &\implies b = n, \end{aligned}$$

Thus we may assume that the signs are all positive, ie

$$n - a = a - b = b - c = c - n.$$

Since

$$(n - a) + (a - b) + (b - c) + (c - n) = 0,$$

it follows that

$$n = a = b = c,$$

and in particular $n = b$.

15. Show that for any positive reals a, b, c,

$$[(a + b)(b + c)(c + a)]^{1/3} \geq \frac{2}{\sqrt{3}}(ab + bc + ca)^{1/2}.$$

Answer:

Method 1 *Suppose a, b, c are the roots of*

$$f(x) = x^3 - Sx^2 + Rx - P.$$

Then

$$S = a + b + c, \ R = ab + bc + ca, \ P = abc.$$

We have

$$\begin{aligned}(a + b)(b + c)(c + a) &= (S - a)(S - b)(S - c) \\ &= f(S) \\ &= RS - P.\end{aligned}$$

Thus we have to show that

$$(RS - P)^2 \geq \frac{2^6}{3^3} R^3$$

ie

$$\Delta = 3^3(RS - P)^2 - 2^6 R^3 \geq 0.$$

Recall the the discriminant of $f(x)$ is

$$D = [(a - b)(b - c)(c - a)]^2.$$

Clearly

$$D \geq 0$$

if the roots of $f(x)$ *are real.*

But suppose $f(x)$ *had one real root* r *and two complex conjugate roots* $s \pm it$*. Then*

$$\begin{aligned} D &= [(r-s-it)(r-s+it)(2it)^2 \\ &= -4t^2[(r-s)^2+t^2]^2. \end{aligned}$$

Thus

$$D < 0.$$

So $D \geq 0$ *is the condition for* $f(x)$ *to have 3 real roots.*

Both D *and* Δ *are symmetric polynomials of order 6 in* a, b, c*. It seems likely that they are equal up to a scalar multiple.*

Method 2 *We use partial differentiation to identify the local minima of*

$$F(a,b,c) = 3^2 f(a,b,c)^2 - 2^6 g(a,b,c)^3,$$

where

$$f(a,b,c) = (a+b)(b+c)(c+a), \; g(a,b,c) = ab+bc+ca,$$

subject to the constraint

$$a+b+c = 3.$$

By the 'Lagrange multiplier' method, at a stationary point

$$\frac{\partial F}{\partial a}da + \frac{\partial F}{\partial b}db + \frac{\partial F}{\partial c}dc = 0$$

whenever

$$da + db + dc = 0.$$

In other words

$$\frac{\partial F}{\partial a} = \frac{\partial F}{\partial b} = \frac{\partial F}{\partial c}$$

at a stationary point.

Now

$$\begin{aligned} \frac{\partial F}{\partial a} &= 3^2 2f\frac{\partial f}{\partial a} - 2^6 3g^2\frac{\partial g}{\partial a}, \\ \frac{\partial F}{\partial b} &= 3^2 2f\frac{\partial f}{\partial b} - 2^6 3g^2\frac{\partial g}{\partial b}, \\ \frac{\partial F}{\partial c} &= 3^2 2f\frac{\partial f}{\partial c} - 2^6 3g^2\frac{\partial g}{\partial c}. \end{aligned}$$

But

$$\frac{\partial f}{\partial a} = (b+c)(2a+b+c), \ \frac{\partial g}{\partial a} = (b+c),$$

so the factor $b+c$ *comes out of the first equation above, and similarly for* $\frac{\partial f}{\partial b}$*, etc.*
After removing these factors, subtraction of the first equation from the second gives

$$(a-b)\ 3^2 2^2 f = 0$$

Thus $a = b$*; and similarly*

$$a = b = c.$$

So $(1,1,1)$ *is the only stationary point, and must therefore give the minimum value 0 to* F*.*

Method 3 *It should simplify matters if we move the origin to* $(1,1,1)$ *(where we are using the same constraint* $a+b+c=3$ *as above.*
So let

$$a = 1+x, \ b = 1+y, \ c = 1+z,$$

where

$$x+y+z = 0.$$

Let x, y, z *be the roots of the cubic*

$$h(t) = (t-x)(t-y)(t-z) = t^3 + rt - p,$$

where

$$r = xy + yz + zx, \ p = xyz.$$

We have

$$\begin{aligned}(a+b)(b+c)(c+a) &= (2+x+y)(2+y+z)(2+z+x) \\ &= (2-z)(2-x)(2-y) \\ &= h(2) \\ &= 2^3 + 2r - p,\end{aligned}$$

while

$$\begin{aligned}ab+bc+ca &= (1+x+y+xy) + (1+y+z+yz) + (1+z+x+zx) \\ &= 3 + r.\end{aligned}$$

It slightly simplifies the calculations if we set $x = 2x',\ y = 2y',\ z = 2z'$. *The equation* $h(t)$ *becomes*

$$j(t) = t^3 + 4rt - 8p,$$

while now

$$(a+b)(b+c)(c+a) = 2^3(1+r-p),$$
$$ab + bc + ca = 3(1+r),$$

and the inequality reads

$$(1+r-p)^2 = (1+r)^3.$$

Challenge Problem

Suppose a, b are two positive integers such that $ab+1$ divides a^2+b^2. Prove that $(a^2+b^2)/(ab+1)$ is a perfect square.

Answer: *Suppose the prime*

$$p \mid ab + 1.$$

Then

$$p \mid a^2 + b^2.$$

In other words,

$$ab \equiv -1 \bmod p,\ a^2 + b^2 \equiv 0 \bmod p.$$

From the first of these,

$$a^2b^2 \equiv 1 \bmod p.$$

Hence a^2, b^2 *are roots of the equation*

$$t^2 - 1 \equiv 0 \bmod p$$

ie

$$(t-1)(t+1) \equiv 0 \bmod p.$$

If $p \neq 2$ *this implies that*

$$a^2 \equiv 1, b^2 \equiv -1 \bmod p \ \textit{or}\ a^2 \equiv -1, b^2 \equiv 1 \bmod p$$

But both these contradict the fact that

$$a^2b^2 \equiv 1 \bmod p.$$

Hence the only prime dividing $ab+1$ is 2. Thus

$$ab + 1 = 2^e.$$

Hence a, b are odd, and so

$$a^2 + b^2 \equiv 2 \bmod 4.$$

Thus

$$e \geq 2 \implies 2^e \nmid a^2 + b^2.$$

It follows that $e = 1$, so the only solution is the trivial one

$$a = b = 1$$