Problem Solving (MA2201)

Week 2

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1. Find the largest positive integer such that $n^3 + 100$ is divisible by $n + 10$.

   Answer: *Let*
   $$N = n + 10.$$
   Then
   $$n^3 + 100 = 0 mod N.$$
   But
   $$n = -10 \bmod N,$$
   so
   $$n^3 + 100 = -900 mod N.$$
   Thus
   $$N \mid 900.$$

2. Find all positive integers n such that $n! + 5$ is a perfect cube.

   Answer: *For any prime p,*
   $$n! + 5 \equiv 5 \bmod p$$
   for all $n \geq p$.

So if $n! + 5$ *is a perfect cube, then 5 is a* cubic residue mod$p$

Recall that $(\mathbb{Z}/p)^\times$ *is a cyclic group of order* $p - 1$*.*

If $3 \nmid p - 1$ *then the homomorphism*

$$x \mapsto x^3 : (\mathbb{Z}/p)^\times \to (\mathbb{Z}/p)^\times$$

has trivial kernel, and so is surjective, ie every element is a cubic residue.

But if $3 \mid p-1$ *then the kernel contains just 3 elements (the elements of order 3), and so* $(p-1)/3$ *elements are cubic residues, so there is a reasonable chance the 5 will not be.*

Take $p = 7$*. There are* $6/3 = 2$ *cubic residues, namely* $\pm 1 \bmod p$*. So 5 is not a cubic residue* $\bmod 7$*, and it follows that* $n! + 5$ *cannot be a cube if* $n \geq 7$*.*

Checking $n! + 5$ *for* $n < 7$ *we see that the only cube occurs when* $n = 5$*.*

3. Show that the product of 4 successive integers cannot be a perfect square.

Answer:

Method 1 *Let the successive integers be* $n, n+1, n+2, n+3$*.*

We have

$$n(n+3) = n^2 + 3n, \; (n+1)(n+2) = n^2 + 3n + 2.$$

Thus

$$n(n+3) < (n+1)(n+2),$$

and so

$$[n(n+3)]^2 < n(n+1)(n+2)(n+3) < [(n+1)(n_2)]^2.$$

Also

$$(n+1)(n+2) = n(n+3) + 2.$$

Thus if $n(n+1)(n+2)(n+3)$ is a perfect square, we must have

$$n(n+1)(n+2)(n+3) = [n(n+3)+1]^2/$$

But that is impossible, since the left-hand side is even while the right-hand side is odd, since one of $n, n+3$ is even.

Method 2 *Suppose $n(n+1)(n+2)(n+3)$ is a perfect square.*

Suppose first that $3 \nmid n$.

Two of the numbers are even, and two are odd. Suppose $n, n+2$ are even. Then

$$\gcd(n, n+1) = 1, \gcd(n, n+2) = 2, \gcd(n, n+3) = 1, \gcd(n+1, n+2)$$

It follows that

$$n = 2a^2,\ n+1 = b^2,\ n+2 = 2c^2,\ n+3 = d^2.$$

But then

$$b^2 < n+3 < (b+1)^2 = n+2+2b.$$

So $n+2$ cannot be a square (unless $b=0$).

In the same way, if $n+1, n+3$ are even then

$$n = a^2,\ n+1 = 2b^2,\ n+2 = c^2,\ n+3 = 2d^2.$$

Again,

$$a^2 < n+2 < (a+1)^2.$$

Now suppose $3 \mid n$. Then $3 \mid n, n+3$.

Suppose $n, n+2$ are even. Then

$$\gcd(n, n+1) = 1, \gcd(n, n+2) = 2, \gcd(n, n+3) = 3, \gcd(n+1, n+2)$$

It follows that

$$n = 6a^2,\ n+1 = b^2,\ n+2 = 2c^2,\ n+3 = 3d^2.$$

Can we go on from here?

4. Find the positive integer n for which

$$[\log_2 1] + [\log_2 2] + [\log_2 3] + \cdots + [\log_2 n] = 2010,$$

where $[x]$ denotes the greatest integer $\leq x$.

Answer: *Let*

$$N(n) = [\log_2 1] + [\log_2 2] + [\log_2 3] + \cdots + [\log_2 n].$$

We have

$$\begin{aligned}
[\log r] &= 0 \ \textit{if}\ r = 1 \\
[\log r] &= 1 \ \textit{if}\ r = 2, 3 \\
[\log r] &= 2 \ \textit{if}\ 4 \leq r < 8 \\
&\ldots \\
[\log r] &= e - 1 \ \textit{if}\ 2^{e-1} \leq r < 2^e.
\end{aligned}$$

Hence

$$\begin{aligned}
N(2^e - 1) &= \sum_1^{2^e - 1} [\log r] \\
&= 0 \cdot 1 + 1 \cdot 2 + 2 \cdot 2^2 + ... + (e - 1)2^{e-1}.
\end{aligned}$$

Let

$$\begin{aligned}
f(x) &= 1 + x + x^2 + ... + x^{e-1} \\
&= \frac{x^e - 1}{x - 1}
\end{aligned}$$

Then

$$\begin{aligned}
f'(x) &= 1 + 2x + ... + (e - 1)x^{e-2} \\
&= \frac{ex^{e-1}}{x - 1} - \frac{x^e - 1}{(x - 1)^2}
\end{aligned}$$

Hence

$$\begin{aligned}
N(2^e - 1) &= 2f'(2) \\
&= 2e2^{e-1} - 2(2^e - 1) \\
&= (e - 2)2^e + 2.
\end{aligned}$$

In particular

$$N(2^9 - 1) = 7 \cdot 2^8 + 2$$

ie

$$N(511) = 1538$$

After this we are adding 8 for each number. Since

$$1538 = 2 \bmod 8,$$

we get numbers = 2 mod 8, such as 2010.

5. If the number n is chosen at random, what is the probability that 2^n ends in 2?

6. If the number n is chosen at random, what is the probability that 2^n starts with 1?

7. Let us say that a number is *almost-prime* if it is not divisible by 2,3,5 or 7. How many almost-prime numbers are there less than 1,000?

   Answer: *Let* $S(r)$ *be the natural numbers* ≤ 1000 *that are divisible by* r*:*

   $$S(r) = \{1 \leq n \leq 1000 : r \mid n\}.$$

   Then

   $$\#S(r) = [1000/r].$$

   We have to determine the number of elements in

   $$X = [1, 1000] \setminus (S(2) \cup S(3) \cup S(5) \cup S(7)).$$

   By the Principle of Inclusion-Exclusion,

   $$\begin{aligned} \#(S(2) \cup S(3) \cup S(5) \cup S(7)) =& \#S(2) + \#S(3) + \#S(5) + \#S(7) \\ & - \#(S(2) \cap S(3)) - \#(S(2) \cap S(5)) - \#(S \\ & + \#(S(3) \cap S(5)) + \#(S(3) \cap S(7)) + \#(S \\ & - \#(S(2) \cap S(3) \cap S(5)) + \#(S(2) \cap S(5) \cap \\ & - \#(S(2) \cap S(3) \cap S(7)) - \#(S(3) \cap S(5) \cap \\ & + \#(S(2) \cap S(3) \cap S(5) \cap S(7) \end{aligned}$$

8. 8 people are sitting round a circular table. In how many ways can they change places so that each person has a different neighbour to the right?

   Answer: *Let us solve the problem with a set X of n people around the table. Let the number of solutions be $f(n)$.*

   For simplicity let us number the people $0, 1, \ldots, n-1 \bmod n$.

   Consider a permutation π of the set X. Suppose in fact just r people still have the same person on their right, Then we can define a permutation σ of the $n-r$ remaining people $X \setminus S$ by identifying i with $i+1$ if

$$\pi(i+1) \equiv \pi(i) + 1.$$

   Thus suppose the interval $[j,k] \subset X$, ie

$$\pi(i+1) \equiv \pi(i) + 1.$$

   for $i = j, j+1, \ldots, j+k$, but

$$\pi(j-1) \not\equiv \pi(j) - 1, \ \pi(k+1) \not\equiv \pi(k) + 1;$$

   and suppose

$$\pi(\ell) = j.$$

   Then we set

$$\sigma(\ell) = k.$$

9. If a set of circles is placed in the plane so that no circle in the set lies inside another one, does it follow that the set is enumerable?

   Answer:

10. Can you find 3integers x, y, z, not all zero, such that

$$x^3 + 2y^3 + 4z^3 = 0?$$

   Answer:

11. What point P in a triangle ABC minimises

$$AP^2 + BP^2 + CP^2?$$

Answer:

12. Show that in a group of 6 people there are either 3 people who know each other ("mututal acquaintances") or 3 people who don't know each other ("Mutual strangers")

Answer:

13. Show that the complex numbers x, y, z form an equilateral triangle if and only if

$$x^2 + y^2 + z^2 = xy + yz + zx.$$

Answer:

14. Show that in any graph with at least 2 vertices there must be 2 vertices with the same degree. (The degree of a vertex is the number of edges with an end-point at that vertex.)

Answer:

15. Show that for any $k > 2$ one can find k integers $0 < a_1 < a_2 < \cdots < a_k$ such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_k} = 1.$$

Answer: *We have*

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1.$$

Suppose

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_k} = 1.$$

Then

$$\frac{1}{k} = \frac{1}{a_k}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right) = \frac{1}{2a_k} + = \frac{1}{3a_k} + = \frac{1}{6a_k},$$

giving a sum of the same kind with 2 additional terms.

The sum

$$\frac{1}{2}+\frac{1}{2}=1$$

gives successive sums with $4, 6, 8, \ldots$ *terms, while*

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1.$$

gives successive sums with $5, 7, 9, \ldots$ *terms.*

Thus we have solutions for all $k \geq 3$.

Challenge Problem

Do there exist primes p, q such that

$$p \mid q(q-1)+1 \text{ and } q \mid p(p-1)+1?$$

Answer: